JOURNAL OF ALGEBRA 73, 434–470 (1981)

Finite Groups Which Possess a Strongly Closed 2-Subgroup of Class at Most Two, I

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Communicated by W. Feit

Received March 12, 1979

1. INTRODUCTION

Suppose G is a finite group and S is a p-subgroup of G where p is a prime number. Then S is said to be a strongly closed p-subgroup of G if S is strongly closed in some Sylow p-subgroup of G (containing S) with respect to G. Let $\mathscr{L}$ denote the class of isomorphism classes of the following groups: $L_2(q)$ (q odd, $q \not\equiv 1(16)$, $q > 3$); a group of Ree type; A_7; J_1; $L_2(2^n)$; $U_3(2^n)$; $\mathrm{Sz}(2^n)$; $L_3(2^n)$; and $PSp_4(2^n)$ ($n \geqslant 2$).

The aim of this paper is to prove

THEOREM A. *Suppose G is a finite non-abelian simple group which contains a non-trivial strongly closed 2-subgroup of nilpotent class at most two. Then $G \in \mathscr{L}$.*

From Theorem A we may obtain several corollaries. (For the definition of $E(G)$ and the term "quasisimple Goldschmidt group" see Section 2.)

COROLLARY 1. *Suppose that G is a finite group, that $O_{2'}(G) = 1$ and that S is a strongly closed 2-subgroup of G of nilpotent class at most two. Set $K = \langle S^G \rangle$. Then $K = HE(K)$ where each of the components of $E(K)$ is of type L for some $L \in \mathscr{L}$ and $H/O_{2',2,2'}(H)$ is a central product of an abelian 2-group and quasisimple Goldschmidt groups.*

Certain methods that have been employed in studying non-abelian finite simple groups frequently lead to configurations which are dominated by the interaction between 2-subgroups and 3-subgroups (see, for example, [21]). In this connection the next result may be of some interest.

0021-8693/81/120434–37$02.00/0

COROLLARY 2. *Let G be a non-abelian finite simple group containing a non-trivial strongly closed 2-subgroup S. Then either*

(i) $G \in \mathscr{L}$; *or*

(ii) *for each* $1 \neq x \in N_G(S)$ *with* $x^3 = 1$, $C_S(x) \neq 1$.

Clearly, the usefulness of Theorem A will depend upon a sufficient supply of "interesting" 2-subgroups of class at most two. As mentioned in [20], in a group of even order a result of Thompson's [12, Theorem 5.3.11] guarantees the existence of many "interesting" 2-subgroups of class at most two. A further scenario in which 2-subgroups of class at most two appear is in certain configurations associated with the failure of the "pushing-up problem" for the prime 2 (see [13, Chap. IV, Sect. 12] for further details).

Of course Theorem A extends the results of both Goldschmidt [10] and Gilman and Gorenstein [5, 6]. Indeed, several of the ideas and techniques used in the proof of Theorem A were also used in [10] and [5, 6]. The "characteristic 2-type part" of Theorem A, contained in [20], follows the approach of [6]. The remainder of the proof of Theorem A, contained in this paper and its sequel, proceeds along the lines of [10] using the Bender method. There are, however, several features of the class two problem which make some of the arguments here more involved than in [10]. Much of the blame for this can be lain at the door of Lemma 3.4, which is not as strong as its counterpart [10, (3.7)]. There are other points of difference between [10] and the present work. For example, lack of appropriate analogues of (2.6) and (2.8) of [10] leads to the lengthy Lemma 5.6 (compare with [10, (5.5)]).

This paper is arranged as follows. In Section 2 we establish notation and list some of the results we shall need. Before beginning the examination of a minimal counterexample to Theorem A in Section 4, we establish certain properties of $\mathscr{L}(S)$-groups in Section 3. Section 5 is devoted to proving some uniqueness theorems, which, when combined with the work of Section 6 on the p-constrained case, leads to the important reduction given in Lemma 7.3. In Part II, the proof of Theorem A is completed by showing that the situations in Lemma 7.3 cannot hold in a minimal counterexample.

2. PRELIMINARY RESULTS AND NOTATION

We begin this section by recalling certain results that we shall need on strongly closed 2-subgroups. Until (2.12), G is assumed to be a group which contains a strongly closed 2-subgroup S. All groups in this paper are assumed to be finite.

(2.1) (i) *If* $N \trianglelefteq G$, *then* SN/N *is a strongly closed* 2-*subgroup of* G/N.

(ii) *If* $H \leqslant G$ *and* S_1 *and* S_2 *are maximal elements, under inclusion of the set* $\mathscr{X}(H) = \{R \mid R \leqslant H$ *and* R *is* G-*conjugate to a subgroup of* $S\}$, *then* S_1 *and* S_2 *are* H-*conjugate and* S_1 *is a strongly closed* 2-*subgroup of* H.

(iii) S *is strongly closed in* P *with respect to* G *for each* 2-*subgroup* P *of* G *containing* S.

(iv) $G = \langle S^G \rangle N_G(S)$.

(v) $N_G(S)$ *controls* G-*fusion in* $C_G(S)$.

Proof. For (i)–(v) consult (2.1)–(2.3) of [11]. Part (iv) may be proved by a Frattini argument.

In the situation of (2.1)(ii) we will use $\mathscr{X}^*(H)$ to denote the maximal elements of $\mathscr{X}(H)$ under inclusion, and we also set $H^* = \langle \mathscr{X}^*(H) \rangle$.

(2.2) *Suppose* $G = SN \trianglerighteq N$, $S \cap N = 1$ *and* $O_{2'}(N) = 1$. *Then* $[S, N] = 1$.

Proof. See [11, (2.4)].

(2.3) [19, Lemma 2.10]. *Suppose* $C_G(O_2(G)) \leqslant O_2(G)$, *and set* $S^* = S \cap O_2(G)$. *Then* $C_S(S^*) \leqslant S^*$.

(2.4) *Either* $\langle C_G(\sigma) \mid \sigma \in S^{\#},\ \sigma^2 = 1 \rangle \geqslant \langle \Omega_1(S)^G \rangle$ *or* $\langle \Omega_1(S)^G \rangle$ *has a strongly embedded subgroup.*

Proof. See [11, (2.7)].

(2.5) (Goldschmidt [11, Corollary B3]). *If* $N_G(S)/C_G(S)$ *is a* 2-*group, then* $S \in \mathrm{Syl}_2\langle S^G \rangle$.

For a group H, H^∞ denotes the terminal member of the derived series.

(2.6) (Goldschmidt [11, Corollary B1]). $C_G(S)^\infty O_{2'}(G) \trianglelefteq G$.

We let Σ denote the set of subgroups of S as defined in [20] prior to Theorem 3.7. Set $\mathscr{F}_0 = \{(S_0, N_G(S_0)) \mid S_0 \in \Sigma\}$.

(2.7) (i) $\mathscr{F}_0$ *is a weak conjugation family for* S.

(ii) *If* $S_0 \in \Sigma$ *and* $S_0 \neq S$, *then, setting* $\bar{N} = N_G(S_0)/S_0$, $\langle \overline{N_S(S_0)}^{\bar{N}} \rangle$ *has a strongly embedded subgroup.*

Proof. See Theorem 3.7 and Lemma 3.8 of [20].

(2.8) [20, Theorem 4.1]). *If* S *has nilpotence class* n *and* $O_{2,2}(G) = 1$, *then the exponent of* $Z(S)$ *is at most* $\max\{2, 2^{n-1}\}$.

We now list some results in which the structure of G is determined from certain given data about S.

Before stating the next theorem we introduce the following terminology: a group H is called a quasisimple Goldschmidt group if $H' = H$ and $H/Z(H)$ is isomorphic to one of the following groups: $L_2(q)$, $\mathrm{Sz}(q)$, $U_3(q)$ $(q = 2^a > 2)$; $L_2(q)$ $(q > 3$ and $q \equiv 3, 5(8))$; J_1; a group of Ree type.

(2.9) (Goldschmidt [10]). *If S is abelian, then $\langle S^G \rangle / O_{2'}(\langle S^G \rangle)$ is isomorphic to a central product of quasisimple Goldschmidt groups and an abelian 2-group.*

(2.10) (J. Hall). *Suppose S is extraspecial, and set $K = \langle S^G \rangle$. Then either*

(i) $Z(S) \leqslant Z^*(G)$; *or*

(ii) *$K/O_{2'}(K)$ isomorphic to one of the following*: S; A_7; $\mathrm{PGL}(2, q)$ $(q \equiv 3, 5(8))$; $L_2(q)$ $(q$ *odd*, $q \not\equiv 1 \pm (16)$, $q > 3)$.

Proof. By combining the main theorem and Theorem 7.2 of [16].

(2.11) (Rowley [20]). *Suppose that G is a non-abelian simple group and that S is non-trivial and has nilpotence class at most two. Furthermore, if for each non-trivial subgroup S_0 of S (setting $H = N_G(S_0)$) $C_H(O_2(H)) \leqslant O_2(H)$, then G is isomorphic to one of the following groups*: $L_2(7)$, $L_2(9)$, $L_2(q)$, $\mathrm{Sz}(q)$, $U_3(q)$, $L_3(q)$, $PSp_4(q)$, *where $q = 2^a > 2$.*

The next result is a slight generalization of [10, Corollary 5]:

(2.12) *Suppose G is a non-abelian simple group and S contains an abelian subgroup A of index 2. Then S' is cyclic.*

Proof. Let $T \in \mathrm{Syl}_2 G$ be such that $S \leqslant T$. By (2.9), $\Omega_1(A)$ is not a strongly closed 2-subgroup of G. Therefore, there exists an involution $\sigma \in S \backslash A$ with σ G-conjugate to an element of A and $S' = [A, \sigma]$. If A is not weakly closed in T with respect to G, then $S = AA^g$ for some $g \in G$. Then $[A : C_A(\sigma)] \leqslant 2$, and so $|[A, t]| \leqslant 2$. Hence we may suppose A is weakly closed in T with respect to G and now appealing to [10, Corollary 4] gives (2.12).

We say that G is an $\mathscr{L}(S)$-group if

(i) S is a strongly closed 2-subgroup of G with S of nilpotent class at most two; and

(ii) if $G \geqslant H \trianglerighteq K$ where $\bar{H} = H/K$ is a non-abelian simple group and $R \in \mathscr{K}^*(H)$, then $\bar{R} \neq 1$ implies that $\bar{H} \in \mathscr{L}$.

We leave strongly closed 2-subgroups for the moment to review some concepts and results that underly the "Bender method."

Let G be a group. A component of G is a subnormal quasisimple subgroup of G. We set $E(G) = \langle K \mid K$ is a component of $G \rangle$ and $F^*(G) = E(G)F(G)$. Note that a component of G is also a component of $E(G)$.

(2.13) (i) *$E(G)$ is a central product of the components of G, which are permuted under conjugation by G.*

(ii) *Suppose $X \leqslant G$ and K is a component of $E(G)$. Then either $K \leqslant [K, X]$ or $[K, X] = 1$. If $X \leqslant N_G(K)$, then either $K \leqslant E(X)$ or $[K, X] = 1$. Also $[E(G), X]$ is the product of those components of $E(G)$ not centralized by X.*

(iii) $C_G(F^*(G)) \leqslant F^*(G)$.

(iv) *If $X \trianglelefteq \trianglelefteq F^*(G)$ and $C_{F^*(G)}(X) \leqslant X$, then $X = E(G)(X \cap F(G))$.*

(v) *If $F^*(G)$ is a p-group, p a prime, and P is a p-subgroup of G, then $F^*(N_G(P))$ and $F^*(C_G(P))$ are also p-groups.*

(vi) *Suppose $O_{2'}(G) = 1$, t is an involution in G and $1 \neq X \leqslant O_{2'}(C_G(t))$. Then there exists a $\langle t, X \rangle$-invariant component of $E(G)$ which is not centralized by either t or X.*

Proof. For parts (i)–(iv) see (2.1) and (2.2) of [10] and for part (v) see (1.8) of [3]. Part (iv) is proved in [11, (2.6)].

(2.14) *Suppose G is a non-abelian simple group, H is a maximal subgroup of G, $X \trianglelefteq \trianglelefteq F^*(H)$ and $C_{F^*(H)}(X) \leqslant X \leqslant M \leqslant G$. Then*

(i) $O_p(M) \cap H = 1$ for $p \notin \pi(F(H))$.

(ii) $[O_p(M), O^p(F^*(H))] = 1$ *for* $p \in \pi(F(H))$.

(iii) *If M is also a maximal supgroup of G and there exists $Y \trianglelefteq \trianglelefteq F^*(M)$ such that $C_{F^*(M)}(Y) \leqslant Y \leqslant H$, then either $H = M$ or $F^*(H)$ and $F^*(M)$ are p-groups for the same prime p.*

Proof. See (1.7) of [3].

It is shown in [14] that a group G possesses a unique normal subgroup, which is minimal subject to covering $E(G/O_{2'}(G))$. This subgroup, which we shall denote by $E_{2'}(G)$, is called the 2-layer of G. Furthermore, the normal subgroups of $E_{2'}(G)$ minimal subject to covering the components of $E(G/O_{2'}(G))$ are uniquely determined (see [14]) and these are referred to as the 2-components of both G and $E_{2'}(G)$.

For a group G, $\mathscr{I}(G)$ denotes the set of involutions contained in G. Let P be a p-group, p a prime. Then cl P denotes the nilpotent class of P and $\mathfrak{A}_e(P)$ is the set of elementary abelian subgroups of P of maximal possible rank. If P is a 2-subgroup of G, then $\mathscr{C}(P) = \{x \in \mathscr{I}(P) \mid E_{2'}(C_G(x)) \neq 1\}$.

(2.15) *Suppose* $x \in \mathscr{I}(G)$ *and* X *is a* $C_G(x)$*-invariant subgroup of* G *with* $X = O_{2'}(X) E(X)$. *Let* K *be a component of* $E(X)$ *and let* L *be a component of* $E(G)$. *Then*

(i) $X \leqslant N_G(L)$.

(ii) *Suppose* $M = LL^x \neq L$. *Then* $[O_{2'}(X), L] = 1$ *and either* $[K, L] = 1$, $K \trianglelefteq M$ *or* $K = \{zz^x \mid z \in L\}$.

(iii) *If* K *is a component of* $C_G(x)$, *then one of the following holds*: $K = L$ *or* $[K, L] = 1$ *or* $L \neq L^x$ *and* $K = C_{[L,x]}(x)'$ *or* $L = [L, x] \geqslant K$.

(iv) *If* $V = \langle x, y \rangle$ *is a non-cyclic subgroup of* G *of order four and* $L^x \neq L$, *then* $LL^x \leqslant \langle C_G(z) \mid z \in V^\# \rangle$.

Proof. For (i) and (ii) see [1, Lemma 2.6], and for (iii) and (iv) see, respectively, Lemmas 2.7 and 2.8 of [1].

We now list some pertinent properties of $\mathscr{L}(S)$-groups. Let $\mathscr{L}_1$ denote the isomorphism classes of the following: $L_2(q)$, q odd, $q \not\equiv \pm 1$ (16), $q > 3$; A_7, J_1, a group of Ree type, and set $\mathscr{L}_2 = \mathscr{L} \backslash \mathscr{L}_1$. Until (2.19) we assume the following situation: G is an $\mathscr{L}(S)$-group for which $\langle S^G \rangle = G$, $O_{2'}(G) = 1$ and $K = F^*(G)$ is a quasisimple group with $K/Z(K) \in \mathscr{L}$. Let $T \in \mathrm{Syl}_2 G$ be such that $S \leqslant T$, and set $R = T \cap K$.

(2.16) (i) *G/K is abelian.*

(ii) $[T : R] \leqslant 2$ *and one of the following holds*: (a) $R = T$; (b) $K \cong L_2(q)$, $q \equiv 3, 5(8)$, $O^{2'}(G) = KT \cong PGL(2, q)$; (c) $K \cong L_2(q^2)$, $q \equiv 3, 5(8)$ *or* $q = 2^n$, $O^{2'}(G) = KT$ *and* $T = R\langle x \rangle$ *where* $x \in \mathscr{I}(T)$ *and induces a field automorphism on* K; (d) $K \cong A_7$ *and* $G \cong S_7$.

(iii) *Either* $S = T$ *or* $S = T' = \Omega_1(T)$, $G = K$ *and* $K/Z(K) \cong \mathrm{Sz}(2^n)$ *or* $U_3(2^n)$ $(n > 1)$.

(iv) *Suppose* $K/Z(K) \in \mathscr{L}_2$. *If* L *is a quasisimple* R*-invariant subgroup of* K *with* $L/Z(L) \in \mathscr{L}$, *then* $L = K$. *Also*, $S \cap K$ *does not normalize any non-trivial subgroups of* K *of odd order.*

(2.17) *Let* $\sigma \in \mathscr{I}(S)$ *and set* $L = E(C_K(\sigma))$. *Then one of the following holds*:

(i) $L = 1$, $K \in \mathscr{L}_1$, *and if* $K \not\cong A_7$, *then* $C_G(\sigma)$ *has a normal 2-complement and a Sylow 2-subgroup of* $C_K(\sigma)$ *is cyclic or dihedral when (respectively)* $\sigma \notin R$ *and* $\sigma \in R$;

(ii) $L = 1$, $K/Z(K) \in \mathscr{L}_2$, *and either* $C_K(\sigma)$ *is 2-closed or* $K \cong PSp_4(2^n)$;

(iii) $\sigma \in Z(K)$ *and* $L = K$;

(iv) *$\sigma \in R$, $K \cong J_1$ or a group of Ree type, $C_K(\sigma) = \langle\sigma\rangle \times L$, where (respectively) $L \cong L_2(5)$ and $L \cong L_2(3^m)$;*

(v) *$\sigma \notin R$, $K \cong L_2(q^2)$, $q \equiv 3, 5(8)$, $C_K(\sigma) \cong PGL(2, q)$ and $L \cong L_2(q)$;*

(vi) *$\sigma \notin R$, $K \cong L_2(2^{2n})$, $n > 1$, and $C_K(\sigma) = L \cong L_2(2^n)$;*

(vii) *$\sigma \notin R$, $K \cong A_7$ and $C_K(\sigma)$ is isomorphic to either S_4 or S_5 (depending on whether σ induces (12) (34) (56) or (12) on K).*

(2.18) (i) *If K is simple, then every involution of K is conjugate in K to an element of $\mathcal{I}(Z(S \cap K))$.*

(ii) *Either K has only one conjugacy class of involutions or $K \cong PSp_4(2^n)$ or $Z(K) \neq 1$ and either $K/Z(K) \cong L_3(4)$ or $\mathrm{Sz}(8)$.*

(iii) *$\mathcal{I}(R) \subseteq \bigcup\{A \mid A \in \mathfrak{A}_e(S)\}$.*

(iv) *If $K/Z(K) \in \mathcal{L}_2$ and $\sigma \in \mathcal{I}(S)$, then $O_{2'}(C_G(\sigma)) = 1$.*

(v) *If $K/Z(K) \in \mathcal{L}_1 \backslash \{L_2(5)\}$ and V is a fours subgroup of S, then $K \leqslant \langle C_G(v) \mid v \in V^{\#}\rangle$.*

(3.19) *Suppose $G = K$ is simple and let $A \in \mathfrak{A}_e(S)$. If $A_0 \leqslant A$ with $[A : A_0] \leqslant 2$, then every involution of K is conjugate to an element of $A_0^{\#}$.*

Proof. By (2.18)(i), (iii) we only need to check the case $K \cong PSp_4(2^n)$ $(n \geqslant 2)$. In this case, $Z(S) = A_1 \times A_2$ $(\leqslant A)$, where $|A_i| = 2^n$ *and* $A_1^{\#}$, $A_2^{\#}$, $Z(S)\backslash(A_1 \cup A_2)$ are the G-conjugacy classes of the involutions of $Z(S)$. Since $n \geqslant 2$, $A_i \cap A_0 \neq 1$ $(i = 1, 2)$ and so (2.19) follows using (2.18)(i).

(2.20) *Suppose G is a non-abelian simple group, H is a maximal subgroup of G and $x \in \mathcal{I}(H)$ with x contained in the centre of some Sylow 2-subgroup of G. If x does not lie in any other conjugate of H, then either G has a strongly embedded subgroup or $G \cong A_n$ (n odd).*

Proof. This is a special case of a theorem proved (independently) by Holt and Smith; see [17].

Throughout this work the theorem of Feit and Thompson [4] will be used implicitly.

3. $\mathcal{L}(S)$-Groups

LEMMA 3.1. *Let G be a group containing a strongly closed 2-subgroup S.*

(i) *For each 2-component L of $K = \langle S^G\rangle$, $L = \langle (S \cap L)^L\rangle$.*

(ii) *Suppose $G = \langle S^G\rangle$, $O_{2'}(G) = 1$, $C_G(S_0) \leqslant H \leqslant G$, where*

$1 \neq S_0 \leqslant S$, *and let* $S \cap H \leqslant R \in \mathscr{X}^*(H)$. *Then every* 2-*component of* H *is contained in* $\langle R^H \rangle = H^*$.

Proof. (i) Set $\bar{K} = K/O_{2'}(K)$. If $\bar{L} = \langle (\overline{S \cap L})^{\bar{L}} \rangle$, then $L = \langle (S \cap L)^L \rangle O_{2'}(L)$ and hence, as L is perfect, $L = \langle (S \cap L)^L \rangle$. Therefore we may suppose that $O_{2'}(K) = 1$. Hence $E(K) = L_{2'}(K)$. Set $M = \langle L \mid L$ is a component of K and $L \cap S \leqslant Z(L) \rangle$. Since S permutes the components of $E(K)$ by conjugation, S normalizes M. Set $\tilde{M} = M/Z(M)$. Suppose $\tilde{S} \cap \tilde{M} \neq 1$. Since $\tilde{M} = \tilde{L}_1 \times \cdots \times \tilde{L}_n$, where each $\tilde{L}_i$ is a simple group, there exists $\tilde{\lambda}_1 \cdots \tilde{\lambda}_n \in \tilde{S}$, $\tilde{\lambda}_i \in \tilde{L}_i$, $\tilde{\lambda}_i^2 = 1$ with, say, $\tilde{\lambda}_1 \neq 1$. Because $\tilde{L}_1$ is simple and $\tilde{S} \cap \tilde{M}$ is a strongly closed 2-subgroup of $\tilde{M}$ there exists $\tilde{l} \in \tilde{L}_1$, such that $\tilde{\lambda}_1^{\tilde{l}} \neq \tilde{\lambda}_1$ and $\tilde{\lambda}_1^{\tilde{l}} \tilde{\lambda}_2 \cdots \tilde{\lambda}_n = (\tilde{\lambda}_1 \cdots \tilde{\lambda}_n)^{\tilde{l}} \in \tilde{S}$. But then $1 \neq \tilde{\lambda}_1 \tilde{\lambda}_1^{\tilde{l}} \in \tilde{S}$, contrary to $\tilde{S} \cap \tilde{L}_1 = 1$. Thus $\tilde{S} \cap \tilde{M} = 1$. Applying (2.2) to $\tilde{S}\tilde{M}$ gives $[S, M] \leqslant Z(M)$, and hence $[S, M] = 1$ by the three subgroups lemma. From (2.6) we obtain $C_K(S)^\infty \trianglelefteq K$ and hence $C_K(S)^\infty \leqslant Z(K)$. Thus $M \leqslant C_K(S)^\infty = 1$. Hence $S \cap L \nleqslant Z(L)$ for all components L of K, so giving $\langle (S \cap L)^L \rangle = L$. This proves (i).

(ii) First we note, as $G = \langle S^G \rangle$ and $O_{2'}(G) = 1$, that $C_G(S)^\infty = 1$ by (2.6). Let $\bar{H}$ denote $H/O_{2'}(H)$, and set $K = \langle R^H \rangle$. Suppose the result is false and, subject to this, choose H so that $|R|$ is maximal. Set $M = \langle L \mid L$ is a 2-component of H, $L \nleqslant K \rangle$. If $\bar{M} = 1$, then for a 2-component L of H, $L \leqslant KO_{2'}(H)$ whence $L = L^\infty \leqslant K$. Thus $\bar{M} \neq 1$. Since $\bar{R} \cap \bar{M} \leqslant \bar{K} \cap \bar{M} \leqslant Z(\bar{M})$ and $\bar{R}$ normalizes $\bar{M}$, (2.2) implies that $[\bar{R}, \bar{M}] = 1$. Because $C_G(R) \leqslant C_G(S_0) \leqslant H$, $(C_M(R))^\infty = J_1 \cdots J_t$ where each J_i is a 2-component of $C_G(R)$. Since $(C_M(R))^\infty \neq 1$ and $C_G(S)^\infty = 1$, $S \neq R$, and so $R_1 > R$ for some $R_1 \in \mathscr{X}^*(N_G(R))$. Therefore, since J_i is a 2-component of $N_G(R)$, $J_i \leqslant \langle R_1^{N_G(R)} \rangle$. Because J_i is a 2-component of $\langle R_1^{N_G(R)} \rangle$, $J_i = \langle (R_1 \cap J_i)^{J_i} \rangle$ by part (i). Hence, since $R \cap J_i = R_1 \cap J_i$, we obtain $J_i = \langle (R \cap J_i)^{J_i} \rangle \leqslant \langle R^H \rangle$, whence $\bar{M} = \bar{J}_1 \cdots \bar{J}_t \leqslant \bar{K}$, a contradiction. Therefore, (ii) holds.

LEMMA 3.2. *Let* G *be a group containing a strongly closed* 2-*subgroup* S. *If* $[S_0, O^2(F^*(G))] = 1$ *for some* $1 \neq S_0 \leqslant S$, *then* $S \cap O_2(G) \neq 1$.

Proof. Suppose $[S_0, O^2(F^*(G))] = 1$ for some $1 \neq S_0 \leqslant S$ but $S \cap O_2(G) = 1$. Since $S \trianglelefteq T$ for some $T \in \mathrm{Syl}_2 G$, $[S, O_2(G)] \leqslant S \cap O_2(G) = 1$, whence $S_0 \leqslant C_G(F^*(G)) = Z(F(G))$, contrary to $S \cap O_2(G) = 1$. Thus the lemma holds.

LEMMA 3.3. *Suppose* G *is an* $\mathscr{L}(S)$-*group, and let* L *be a* 2-*component of* $K = \langle S^G \rangle$.

(i) $[S: N_S(L)] \leqslant 2$ *and if* L *has non-abelian Sylow* 2-*subgroups, then* $S = N_S(L)$.

(ii) *If* $A \in \mathfrak{A}_e(S)$, *then* $A \leqslant N_G(L)$.

Proof. (i) We may, without loss, suppose $O_{2'}(G)=1$. By Lemma 3.1(i), $R=S\cap L\not\leqslant Z(L)$. Let $\sigma\in S\backslash N_S(L)$. Let $\sigma\in S\backslash N_S(L)$ with $\sigma^2\in N_S(L)$. Then L^σ is a component of G and $L\neq L^\sigma$. Now $[R,\sigma]\leqslant LL^\sigma$ and $[R,\sigma]$ is not contained in either L, L^σ or $Z(LL^\sigma)$. However $[R,\sigma]\leqslant S'\leqslant Z(S)$ and so, as S permutes the components of G, S normalizes LL^σ. Therefore, $[S:N_S(L)]\leqslant 2$. Suppose $N_S(L)<S$. Then, for $\rho\in R$, R centralizes ρ^σ and so, since $\rho^{-1}\rho^\sigma\in Z(S)$, R centralizes ρ^{-1}. Hence R is abelian. Let $P\in \mathrm{Syl}_2 LL^\sigma\langle\sigma\rangle$ with $P\geqslant RR^\sigma\langle\sigma\rangle$. Suppose $P\cap L$ is non-abelian. Note that, as $L/Z(L)\in\mathscr{L}$, R is an elementary abelian 2-subgroup. Let $x\in P\backslash R$ be such that $x^2\notin Z(L)$. Then $[x,\sigma]$ has order 4, and hence $[x,\sigma]\notin RR^\sigma$. However $S\cap LL^\sigma\langle\sigma\rangle$ is a strongly closed 2-subgroup of $LL^\sigma\langle\sigma\rangle$, and so $RR^\sigma\langle\sigma\rangle\trianglelefteq P$. Thus $[\sigma,P]\leqslant LL^\sigma\cap RR^\sigma\langle\sigma\rangle=RR^\sigma$, a contradiction. Therefore, if P is non-abelian, then $S=N_S(L)$.

(ii) In view of part (i), the proof of [5, (2.37)(i)] will also establish (ii).

LEMMA 3.4. *Let G be an $\mathscr{L}(S)$-group. Suppose $\sigma\in\mathscr{I}(S)$ and $X=E(X)\,O_{2'}(F(X))$ is a $C_G(\sigma)$-invariant subgroup of G. Set $Y=O_{2'}(F(X))$.*

(i) *$[X,\sigma]\leqslant F^*(G)$ and $[X,\sigma]=X_1X_2$ where $X_1\trianglelefteq\trianglelefteq F^*(G)$ and $X_2=K_1\times\cdots\times K_m$ with $K_i\cong A_6$ for $i=1,\dots,m$. For each K_i, $K_i<L_i$, where L_i is a component of G and $L_i\cong A_7$. Further $X_2\leqslant\langle S^G\rangle$ and $C_{K_i}(\sigma)=C_{L_i}(\sigma)\cong S_4$ (σ induces* (12) (34) (56) *upon $L_i\cong A_7$).*

(ii) *Suppose K is a component of X for which $[K,O_{2'}(G)]=1$. Then there exists a component L of G such that either $K\leqslant L$ or $K\leqslant LL^\sigma$, where $L\neq L^\sigma$ and the Sylow 2-subgroups of L are abelian. Furthermore, in the former case, $K=L$ or else one of the following holds*:

(a) $K\cong L_2(q)$ ($q=3,5(8)$, $q>3$), L of type JR, $[K,\sigma]=1$ and $\sigma\in LC_G(L)$;

(b) *$K\cong L_2(q)$ ($q=3,5(8)$, $q>3$), $L\cong L_2(q^2)$, $[K,\sigma]=1$ and σ induces a field automorphism on L*;

(c) *$K\cong L_2(2^n)$ ($n\geqslant 2$), $L\cong L_2(2^{2n})$, $[K,\sigma]=1$ and σ induces a field automorphism on L*;

(d) *$K\cong A_5$, $L\cong A_7$, $[K,\sigma]=1$ and σ induces* (12) *upon L*; *and*

(e) *$K\cong A_6$, $L\cong A_7$, $[K,\sigma]\neq 1$ and σ induces* (12) (34) (56) *upon L.*

(iii) *$Y\leqslant\langle S^G\rangle O_{2'}(G)$. Also, for each component L of G, either $[Y,L]=1$ or $L\leqslant\langle S^G\rangle$, L is isomorphic to either A_7 or $L_2(q)$ ($q\not\equiv\,|\pm(16)$, q odd, $q>3$), $Y\leqslant LC_G(L)$ and $Y/C_Y(L)$ is cyclic.*

Proof. We first establish part (i), and will argue by induction upon

$|G|+|X|$. Note that $[X, \sigma]=X_1=E(X_1)\,O_{2'}(F(X_1))$ is a $C_G(\sigma)$-invariant subgroup and so we may suppose that $X=[X, \sigma]$. As in [10, (3.7)] we may reduce to the situation $O_{2'}(G)=1$.

Case 1: G is 2-constrained

Since $[X, C_{O_2(G)}(\sigma)] \leqslant X \cap O_2(G) \leqslant O_2(X) \leqslant Z(X)$, $C_{O_2(G)}(\sigma)$ centralizes Y, and using the "three subgroups lemma," also centralizes $E(X)$. Thus $[X, C_{O_2(G)}(\sigma)]=1$. If $H=O_2(G)X\langle\sigma\rangle \neq G$, then, by induction, since H is an $\mathscr{S}(R)$-group, where $R \in \mathscr{X}^*(H)$, $X=[X, \sigma] \leqslant F^*(H)$. Because $X=O^2(X)$, this gives $[X, O_2(G)]=1$, whence $X=1$. Therefore, we may assume that $O_2(G)X\langle\sigma\rangle=G$.

Set $S^*=S \cap O_2(G)$. By (2.3) $Z(S) \leqslant S^*$. Note that $S^* \trianglelefteq G$. It is claimed that there exists a subgroup N of S such that $Z(S) \leqslant N \trianglelefteq G$ and $[N, X]=1$. If $S^*X\langle\sigma\rangle \neq G$, then by induction $X=[X, \sigma] \leqslant F^*(S^*X\langle\sigma\rangle)$ and so $[X, S^*]=1$. Thus, in this case, we may take $N=Z(S^*)$. If $S^*X\langle\sigma\rangle=G$, then, since $[C_{O_2}(G)(\sigma), X]=1$, $Z(S) \leqslant Z(G)$, and so we may take $N=Z(S)$. Because X has even index in G and $[N, X]=1$, G/N is 2-constrained and $O_{2'}(G/N)=1$. Hence, by (2.3), $S \trianglelefteq G$, whence $[X, O_2(G)]=1$ by $[C_{O_2(G)}(\sigma), X]=1$ and the "$P \times Q$ lemma." Therefore $X=1$, and this settles case 1.

Case 2: G is not 2-constrained

So $E(G) \neq 1$. Note that $F^*(G)=O_2(G)\,E(G)$ and that any component of X is a component of $XC_G(\sigma)$.

(3.1) *We may suppose* $G=\langle X^G\rangle\langle\sigma\rangle$.

Set $N=\langle X^G\rangle$ and suppose $N\langle\sigma\rangle \neq G$. Then induction yields $X=[X, \sigma] \leqslant F^*(N\langle\sigma\rangle)$, with $X=X_1X_2$, $X_1 \trianglelefteq \trianglelefteq F^*(N\langle\sigma\rangle)$, $X_2=K_1 \times \cdots \times K_n$, $K_i \cong A_6$ and each $K_i<L_i \cong A_7$, where L_i is a component of $N\langle\sigma\rangle$. Since $X=O^2(X)$, $X \leqslant F^*(N) \trianglelefteq F^*(G)$, and so the lemma holds. So we may take $G=N<\sigma>$.

(3.2) *If K is a component of X, we may assume K is not a component of G.*

Suppose K were a component of G. Then $\tilde{X}=\langle\tilde{K} \mid \tilde{K}$ is a component of X but not a component of $G\rangle \neq E(X)$. Since $C_G(\sigma)$ will permute the components of X and $E(G) \trianglelefteq G$, $C_G(\sigma)$ normalizes $\tilde{X}$. By induction $\tilde{X}Y=[\tilde{X}Y, \sigma] \leqslant F^*(G)$ and $\tilde{X}Y=\tilde{X}_1\tilde{X}_2$, where $\tilde{X}_1=[\tilde{X}_1, \sigma] \trianglelefteq \trianglelefteq F^*(G)$ and $\tilde{X}_2=\tilde{K}_1 \times \cdots \times \tilde{K}_n$. Also $A_6 \cong \tilde{K}_i<L_i \cong A_7$, where L_i is a component of G. Then $X=[X, \sigma]$ satisfies the conclusions of part (i).

We next show that

(3.3) *$L \trianglelefteq G$ for each component L of G.*

By (2.15)(i), $X \leqslant N_G(L)$ for each component L of G. Set $N = \{g \in G \mid L^g = L$ for all components L of $G\}$. From (3.1) $G = N\langle\sigma\rangle$.

Let L be a component of G and suppose $L \neq L^\sigma$. Then $[L, Y] = 1$ (and hence $[L^\sigma, Y] = [L^\sigma, Y^\sigma] = 1$) by (2.15)(ii). If K is any component of X, then by (2.15)(ii) either $[K, L] = 1$, $K = L$, $K = L^\sigma$ or $K = \{zz^\sigma \mid z \in L\}$. The last possibility contradicts $X = [X, \sigma]$, while (3.2) rules out $K = L$ and $K = L^\sigma$. Therefore, $[K, L] = 1$ for each component K of X, and so $[L, X] = 1$. Clearly $[L^\sigma, X] = [L^\sigma, X^\sigma] = 1$. Thus X centralizes $\tilde{E} = \langle L \mid L$ is a component of G and $L \neq L^\sigma\rangle$.

Because $G = N\langle\sigma\rangle$ we have $\tilde{E} \trianglelefteq G$ and hence $X \leqslant C_G(\tilde{E}) \trianglelefteq G$. By (3.1) $G = C_G(\tilde{E})\langle\sigma\rangle$ whence $\tilde{E} = O^2(\tilde{E}) \leqslant C_G(\tilde{E})$, from which we infer that $\tilde{E} = 1$. Consequently $\sigma \in N$, and so $G = N$. This proves (3.3).

(3.4) *Let L and K be components, respectively, of G and X. If $K \leqslant L$, then either* (i) *holds or $X = K$.*

Suppose $K \leqslant L$, and set $\tilde{X} = Y < \tilde{K} \mid \tilde{K}$ is a component of X and $\tilde{K} \neq K\rangle$. If $X \neq K$, then $|K| < |X|$ and $|\tilde{X}| < |X|$ whence part (i) holds by induction. This verifies (3.4).

(3.5) (i) *$G = LX\langle\sigma\rangle$ where L is a component of G.*

(ii) *$[L, \sigma] = L$.*

(iii) *L is a quasisimple group of type $\mathscr{L}$.*

(iv) *$E(X) \leqslant LC_G(L)$.*

(i) Let L be a component of G. If $LX\langle\sigma\rangle \neq G$, then using induction gives $X = [X, \sigma] \leqslant F^*(LX\langle\sigma\rangle)$ with $X = X_1X_2$ as in the conclusion of (i). Thus either $X \leqslant C_G(L)$ or $L \cong A_7$ and L contains a component of X. In the former case, by (3.1) and (3.3), $G = C_G(L)\langle\sigma\rangle$ whence $L = 1$, a contradiction, while the latter stituation implies $X \leqslant L$ by (3.4). Since $L \trianglelefteq G$, $G = L\langle\sigma\rangle$ by (3.1), contrary to the supposition $LX\langle\sigma\rangle \neq G$. This proves (3.5)(i).

(ii) Suppose $[L, \sigma] \neq L$. Then $L \leqslant C_G(\sigma)$, and so L normalizes X. Thus $[L, Y] = 1$ by (2.13). By (2.13) and (3.2), $[L, E(X)] = 1$, and so $[L, X] = 1$. From (3.1) and (3.3), $G = C_G(L)\langle\sigma\rangle$, which is impossible. Therefore, $L = [L, \sigma]$.

(iii) From (ii), $L \leqslant \langle S^G\rangle$, and so L is a component of $\langle S^G\rangle$. Lemma 3.1(i) implies that $(S \cap L)/Z(L)$ is non-trivial and so, as G is an $\mathscr{L}(S)$-group, L is of type $\mathscr{L}$.

(iv) By (iii), Out L is soluble, and hence $E(X) \leqslant LC_G(L)$.

(3.6) (i) *$C_L(\sigma)$ normalizes $X \cap L$.*

(ii) $S(X \cap L) = F(X \cap L)$.

Since $C_G(\sigma)$ normalizes X, clearly $C_L(\sigma)$ normalizes $X \cap L$. By (3.3), $L \trianglelefteq G$, and so $X \cap L \trianglelefteq X$. Thus $S(X \cap L) \leqslant S(X) = F(X)$, so giving (ii).

(3.7) *Suppose $C_L(\sigma)$ is a maximal subgroup of L. Then the following hold:*

(i) *$X \cap L \leqslant C_L(\sigma)$; and*

(ii) *if J is a component of $C_L(\sigma)$, then $J \nleqslant X$.*

(i) If $X \cap L \nleqslant C_L(\sigma)$, then, as $C_L(\sigma)$ normalizes $X \cap L$, $L = (X \cap L)\, C_L(\sigma) \trianglerighteq X \cap L$. Therefore, either $L \leqslant X$ or $X \cap L \leqslant Z(L)$. The former possibility is ruled out by (3.2) and the latter yields $[L, \sigma] = 1$, against (3.5)(ii). Hence $X \cap L \leqslant C_L(\sigma)$.

(ii) Suppose $J \leqslant X$. Then $J \leqslant X \cap L$ and so, by (i), J is a component of $X \cap L$. Because $X \cap L \trianglelefteq X$ we deduce that J is a component of X. Now (3.4) gives $X = J$, contrary to $X = [X, \sigma] \neq 1$. Therefore, we must have $J \nleqslant X$.

By (2.17) and (3.5)(ii), (iii) either $\sigma \in L$ or $\sigma \notin L$ and L is isomorphic to one of the following groups: $L_2(q)$ ($q \equiv 3, 5(8)$, $q > 3$); $L_2(q^2)$ ($q \equiv 3, 5(8)$); $L_2(2^{2n})$ ($n > 1$); and A_7.

Subcase 2(a): $\sigma \notin L$. We examine, in turn, the possibilities listed above for L, beginning with $L \cong L_2(2^{2n})$, $n > 1$.

By (2.17), $C_L(\sigma) \cong L_2(2^n)$, and by [18, p. 213], $C_L(\sigma)$ is a maximal subgroup of L. Since $n > 1$, $C_L(\sigma)$ is simple and so $X \cap L = 1$ by (3.7). Hence $[X, C_L(\sigma)] \leqslant X \cap L = 1$. Since $\operatorname{Out} L_2(2^{2n})$ is abelian, $X = [X, \sigma] \leqslant LC_G(L)$. Let $x \in X$. Then $x = lc$, where $l \in L$ and $c \in C_G(L)$. Clearly $l \in C_L(C_L(\sigma))$ and so, since $C_L(C_L(\sigma)) = 1$, we have $x \in C_G(L)$. Thus $X \leqslant C_G(L)$ which by (3.1) and (3.3) is impossible. This disposes of the possibility $L \cong L_2(2^{2n})$, $n > 1$.

For $L \cong L_2(q^2)$ ($q \equiv 3, 5(8)$) we have $C_L(\sigma) \cong PGL(2, q)$ is a maximal subgroup of L by (2.17) and [18, p. 213]. Since $\operatorname{Out} L_2(q^2)$ is abelian, the same argument as for $L_2(2^{2n})$ ($n > 1$) may be used except for the case $q = 3$. Now consider $L \cong L_2(9)$. By (3.6) and (3.7), $L \cap X$ must be a nilpotent normal subgroup of $PGL(2, 3) \cong S_4$ and so $L \cap X$ is a 2-group. Thus $[Y, C_L(\sigma)] \leqslant Y \cap L = 1$ and $[E(X), C_L(\sigma)] \leqslant E(X) \cap L \leqslant Z(E(X))$. Hence $[X, C_L(\sigma)] = 1$, from which we conclude that $X \leqslant C_G(L)$, a contradiction.

Now consider the case $L \cong L_2(q)$ ($q \equiv 3, 5(8)$, $q > 3$). Then $C_L(\sigma)$ is a dihedral group of twice odd order. Also, $C_L(\sigma)$ is a maximal subgroup of L [18, p. 213] and $\operatorname{Out} L_2(q)$ is abelian. From (3.5)(ii) and (3.7)(i), $X \cap L \leqslant O_{2'}(C_L(\sigma))$. Therefore, $[E(X), C_L(\sigma)] = 1$. Let μ be an involution in $C_L(\sigma)$.

Then as $[Y, \langle\mu\rangle] \leqslant Y \cap L$, $Y = (Y \cap L)\, C_Y(\mu)$. Since $C_Y(\mu)$ is σ-invariant and $Y \cap L \leqslant C_L(\sigma)$, $Y = [Y, \sigma] \leqslant C_Y(\mu)$. Because μ inverts $Y \cap L$, we infer that $Y \cap L = 1$. Hence $[Y, C_L(\sigma)] = 1$, and so $[X, C_L(\sigma)] = 1$. Consequently, since $C_L(C_L(\sigma)) = 1$, $X \leqslant C_G(L)$, so dealing with $L \cong L_2(q)$ $(q = 3, 5(8), q > 3)$.

Now suppose $L \cong A_7$. Then $C_L(\sigma) \cong S_5$ or S_4. If $C_L(\sigma) \cong S_5$, then $C_L(\sigma)$ is a maximal subgroup of L, and this possibility may be eliminated as for the case $L \cong L_2(q^2)$. Therefore, $C_L(\sigma) \cong S_4$ (and we may suppose that σ induces (12) (34) (56) upon L). Note that $C_L(\sigma) \leqslant K$, where K is isomorphic to A_6 and is the unique maximal subgroup of L containing $C_L(\sigma)$. Since $L \not\leqslant X$ and K is simple, either $X \cap L \leqslant C_L(\sigma)$ or $X \cap L = K$. Suppose $X \cap L \leqslant C_L(\sigma)$ holds, then by (3.6), $X \cap L$ is a 2-group, and so $[X, C_L(\sigma)] = 1$. Because $X \leqslant LC_G(L)$, this gives the untenable $X \leqslant C_G(L)$. Hence $X \cap L = K$ must hold. Then (3.4) implies that either (i) holds or $X = K$, whence (i) holds.

Subcase 2(b): $\sigma \in L$. Since $L \trianglelefteq G$, $X = [X, \sigma] \leqslant L \in \mathscr{L}$. If L is of one of the following types $L_2(2^n)$; $\mathrm{Sz}(2^n)$; $L_3(2^n)$; $U_3(2^n)$; $PSp_4(2^n)$ $(n \geqslant 2)$, then, as σ is a central involution and G is an $\mathscr{L}(S)$-group, by (2.16)(iv) and (2.17), $X = 1$ or $X = L$ and we are done. If L is of type JR or $L_2(q)$ (q odd, $q \not\equiv \pm 1(16)$, $q > 3$), then $C_G(\sigma)$ being a maximal subgroup implies that either $X = 1$ or $X = L$ (note that, by (3.6)(ii), $L \cong SL(2, q)$, $q = 3, 5(8)$ is not possible). So we are left with the case $L \cong A_7$. The structure of A_7 then forces $X \leqslant C_G(\sigma)$, whence $X = 1$.

This completes the proof of part (i).

We now prove (ii). If $K \leqslant [X, \sigma]$, then (ii) follows from (i). So we may suppose $[K, \sigma] = 1$. If $K \leqslant LL^\sigma$ does not hold for some component L of G, then (2.15)(iii) implies that $[K, E(G)] = 1$. Since $[K, O_{2'}(G)] = 1$ and, by the $P \times Q$ lemma, $[O^2(C_X(\sigma)), O_2(G)] = 1$, $K \leqslant C_G(F^*(G)) = Z(F^*(G))$, a contradiction. Therefore, either $K \leqslant L = L^\sigma$ or $K \leqslant LL^\sigma$, where $L \neq L^\sigma$ and $K = (C_{[L,\sigma]}(\sigma))'$ for some component of L of G. In the latter case $S \cap L$ is abelian by Lemma 3.3(i). If $K < L = L^\sigma$, then $L = [L, \sigma] \leqslant \langle S^G \rangle$. Hence, by Lemma 3.1, $S \cap L \not\leqslant Z(L)$ and so L is of type $\mathscr{L}$. Using the structure of quasisimple $\mathscr{L}$-groups we obtain possibilities (a)–(e).

And finally we consider (iii). From (2.15)(i), $Y \leqslant N_G(L)$ for each component L of G. Thus $C_Y(LO_{2'}(G)/O_{2'}(G)) = C_Y(L)$. So we may suppose $O_{2'}(G) = 1$. If $L \neq L^\sigma$, L a component of G, then $[L, Y] = 1$ by (2.15)(ii). So we may suppose $G = L\langle\sigma\rangle Y$ where L is a component of G. Further, we may assume $[L, \sigma] = L$ and, by (i), that $Y \leqslant C_G(\sigma)$. If L is not of type $L_2(q)$ (q odd, $q \not\equiv \pm 1(16)$) with σ not acting as a field automorphism or A_7, then $Y \leqslant LC_G(L)$, $Y \cap L = 1$, whence $[Y, L] = 1$, as required. For $L \cong A_7$, we obtain $Y \leqslant LC_G(L)$ with $|Y \cap L| = 3$ or 1. Thus (iii) holds for $L \cong A_7$.

So it remains to deal with the cases $\langle\sigma\rangle L \cong L_2(q)$ or $PGL(2, q)$ (q odd). Let $q = p^m$ and $q_0 = p^n$ where $m = 2^a n$ and $2 \nmid n$. Then, since we aim to show

$Y \leqslant LC_G(L)$, we may suppose LY is embedded in $L\langle\alpha\rangle$ where α is the 2^ath power of the field automorphism. By replacing α by one of its conjugates we may suppose there exists a non-cyclic subgroup A of order four with $A \leqslant C_{L\langle\sigma\rangle\langle\alpha\rangle}(\alpha) \cap C_{L\langle\sigma\rangle}(\sigma)$. Because $C_{L\langle\sigma\rangle}(A) = A$, $C_{L\langle\sigma\rangle\langle\alpha\rangle}(A) = Ax\langle\alpha\rangle$. Now $Y = [Y, A]\, C_Y(A) = (L \cap Y)\, C_Y(A)$. Suppose $Y \not\leqslant L$. Then $C_Y(A) = \langle\beta\rangle$, where β is a power of the field automorphism, and we may further suppose that β has prime order r. Let $m = n_1 r$, $q_1 = q^{n_1}$ and $L_1 = C_L(\beta) \cong L_2(q_1)$. Since $q = q_1^r$ and $2 \nmid r$, $q_0 \equiv q_1 \equiv \delta(4)$ where $\delta = \pm 1$. Because $Y \leqslant C_G(\sigma)$, $[\langle\beta\rangle, O^r(C_L(\sigma))] = 1$ and hence, since β induces an automorphism of order r on the cyclic r-group, $r \neq 2$, $|C_L(\sigma)\colon C_{L_1}(\sigma)| = r$ or $2r$ if $\sigma \in L$ and $=r$ if $\sigma \notin L$ ($\sigma \notin L$ implies $q \equiv 3, 5(8)$).

If $\sigma \in L$, then $[C_L(\sigma)\colon C_{L_1}(\sigma)] = sr$ where $s = 1$ or 2, and so $(q-\delta)/(q_1-\delta) = sr$. Hence $q_1^r - \delta = sr(q_1 - \delta)$ and thus $q_1 \mid sr - 1$ and, using Fermat's little theorem, $q_1 \equiv \delta(r)$. So $q_1 < sr$ and hence $r \mid q_1 - \delta \leqslant sr$. Therefore, either $r = q_1 - \delta$ or $sr = q_1 - \delta$. In either case, as $q_1 = \delta(4)$, this forces r to be even, a contradiction.

Now suppose $\sigma \notin L$. Then $|C_L(\sigma)| = q + \delta$, where $q = \delta(4)$, $\delta = \pm 1$. Note that σ acts upon L_1. Thus $|C_{L_1}(\sigma)| = q_1 + \delta$ or $q_1 - \delta$. Suppose $|C_{L_1}(\sigma)| = q_1 + \delta$. Then $q_1^r + \delta = q + \delta = r(q_1 + \delta)$ so giving $q_1 \mid r - 1$ and $r \mid q_1 + \delta$ which implies $r = q_1 + \delta$. Since $2 \nmid r$, this is not possible. So $|C_{L_1}(\sigma)| = q_1 - \delta$ and then we have $q_1^r + \delta = r(q_1 - \delta)$, which gives $q_1 \mid r + 1$. Combining this with $r \mid q_1 + \delta$, we see that $r \leqslant q_1 + \delta \leqslant r + 2$ whence, as r is odd, $r = q_1 + \delta$, a contradiction. Thus we conclude that $Y \leqslant LC_G(L)$. To complete the proof of (iii) we show that $Y \leqslant \langle S^G\rangle O_{2'}(G)$. Without loss of generality we may assume $O_{2'}(G) = 1$. So $Y \leqslant C_G(\sigma)$ by (i). By the above and the $P \times Q$ lemma, Y induces inner automorphisms upon $F^*(G)$ and hence $Y \leqslant F^*(G)$. Let E be the product of the components of G not contained in $\langle S^G\rangle$. Then $Y \leqslant C_G(E) \cap F^*(G) \leqslant O_2(G)\langle S^G\rangle$, whence $Y \leqslant \langle S^G\rangle$, as required.

LEMMA 3.5. *Suppose G is an $\mathscr{L}(S)$-group and $m(O_p(G)) \leqslant 2$ for some odd prime p. Set $G_0 = G^*O_{2'}(G)$ and $\tilde{G}_0 = G_0/C_{G_0}(O_p(G))$. Then*

(i) *$S(\tilde{G}_0)$ is p-closed and either $S(\tilde{G}_0) = \tilde{G}_0$ or $(\tilde{G}_0/(O_p(\tilde{G}_0))^\infty \cong SL(2, p)$ with $p > 3$ and $p \nmid [\tilde{G}_0 : \tilde{G}_0^\infty]$; and*

(ii) *let $P_0 \in \mathrm{Syl}_p\, C_{G_0}(O^p(F^*(G)))$. Then either $O_p(G) = P_0$ or $[P_0 : O_p(G)] = p$, $(\tilde{G}_0/O_p(\tilde{G}_0))^\infty \cong SL(2, p)$, $p > 3$ and $\langle P_0^{G_0}\rangle$ covers $(\tilde{G}_0/O_p(\tilde{G}_0))^\infty$.*

Proof. Set $P = O_p(G)$, $P_0 = \Omega_1(Z_2(P))$, $X = C_G(P)$, $Y = C_G(P_0)$ and $Z = C_{G_0}(O^p(F^*(G)))$. If P is non-cyclic, then $m(P_0) = 2$, Y/X is a p-group, G/Y is isomorphic to a subgroup of $GL(2, p)$ and any subgroup of G/Y is either p-soluble and p-closed or contains $SL(2, p)$. So G_0Y/Y is either p-

soluble and p-closed or contains $SL(2, p)$. Suppose the latter were to hold, and let H denote the inverse image in G_0 of the subgroup of G_0Y/Y isomorphic to $SL(2, p)$. Then, since $\langle S^{G^*}\rangle = G^*$ by (2.1)(iv) and $[G_0 : H]$ is odd, $G_0 = HO_{2'}(G)$ and, furthermore, we must have $p > 3$. Thus $(G_0Y/Y)^{\infty} \cong SL(2, p)$, $p > 3$, and since $G_0 \cap Y/G_0 \cap X$ is a p-group, an isomorphism theorem yields (i) when P is non-cyclic. If P is cyclic, then $\tilde{G}_0$ is abelian, and so we have proved (i).

We now prove (ii). From (i), $\tilde{P}_0 \cap S(\tilde{G}_0) \leqslant O_p(\tilde{Z})$. Let $P_1 = P_0 \cap K$, where K is the inverse image in Z of $\tilde{P}_0 \cap S(\tilde{G}_0)$. Then, since $\tilde{Z} \cong Z/Z \cap X$, $Z \cap X \leqslant Z(F(G))$ by (2.13) and $[P_0, O^p(F(G))] = 1$, we obtain $P_1 \leqslant O_p(G)$. Hence by (i) either $O_p(G) = P_0$ or $[P_0 : O_p(G)] = p$ and $(\tilde{G}_0/O_p(\tilde{G}_0))^{\infty} \cong SL(2, p), p > 3$. Clearly $\langle P_0^{G_0}\rangle$ covers $(\tilde{G}_0/O_p(\tilde{G}_0))^{\infty}$. This establishes (ii).

LEMMA 3.6. *Suppose G is an $\mathscr{L}(S)$-group, p is an odd prime and let R be an elementary abelian subgroup of S with $m(R) \geqslant 3$. Then $\bigcap_{\rho \in R^{\#}} O_p(C_G(\rho)) \leqslant O_p(G)$.*

Proof. Set $Q = \bigcap_{\rho \in R^{\#}} O_p(C_G(p))$. We may suppose that $O_p(G) = 1$ and prove that $Q = 1$. Using the $P \times Q$ lemma we obtain $[O_2(G), Q] = 1$ and from $O_{2'}(F(G)) = \langle O_{2'}(F(G)) \cap C_G(\rho) \mid \rho \in R^{\#}\rangle$ we see that $[O_{2'}(F(G)), Q] = 1$. If $[F^*(G), Q] = 1$, then $Q = 1$, and we are done. So $[L, Q] \neq 1$ for some component L of $E(G)$. Hence $L \cong A_7$ or $L_2(q)$ some odd q by Lemma 3.4(iii). Also $R \leqslant N_G(L)$ by (2.15)(ii). If $L \leqslant C_G(\rho)$ for some $\rho \in R^{\#}$, then $[L, O_p(C_G(\rho))] = 1$ by (2.13), against $[L, Q] \neq 1$. Therefore, $R \cap L$ is non-cyclic and so $L \leqslant \langle C_G(\rho) \mid \rho \in R^{\#} \rangle$, which also yields $[L, Q] = 1$. This establishes the lemma.

LEMMA 3.7. *Suppose G is a group with a strongly closed 2-subgroup S, $O_{2'}(G) = 1$, $G = \langle S^G\rangle$ and* cl $S \leqslant 2$. *Then*

(i) *$G = HE(G)$ where $S \leqslant H$, $H = \langle S^H\rangle$ and H is 2-constrained; and*

(ii) *if G is an $\mathscr{L}(S)$-group, then $O_2(G) \leqslant S$.*

Proof. By a Frattini argument $G = NE(G)$, where $N = N_G(Q)$ and $Q \in \mathrm{Syl}_2\, O_2(G)\, E(G)$. Clearly we may choose Q such that $S \leqslant N$. Set $H = \langle S^N\rangle$. Then $\langle S^H\rangle = H$ and, since $S \leqslant HE(G) \trianglelefteq G$, $G = HE(G)$. By [15, Proposition 3.3] N is 2-constrained whence H is 2-constrained, and so we have proved (i).

Since $G = HE(G)$ and $N \geqslant H$, $N = H(N \cap E(G))$. So, as $O_2(G) \leqslant N$,

$$O_2(G) = (O_2(G) \cap H)(O_2(G) \cap E(G)) \leqslant O_2(H)\, O_2(E(G)).$$

By (2.3) and part (i), $O_2(H) \leqslant S$, and, by (2.16)(iii) and Lemma 3.1, $O_2(E(G)) \leqslant S$. Therefore, (ii) holds.

For each p-group P, p a prime, Glauberman [8, p. 37] has defined a characteristic suubgroup $K_\infty(P)$. Properties and results concerning $K_\infty(P)$ that we will need are tabulated in the next lemma.

LEMMA 3.8 (Glauberman). *Let P be a p-group, p a prime.*

(i) $K_\infty(P)\operatorname{ch} P$ *and if* $P \neq 1$, *then* $K_\infty(P) \neq 1$.

(ii) *If* $K_\infty(P) \leqslant Q \leqslant P$, *then* $K_\infty(P) = K_\infty(Q)$.

(iii) *Suppose G is a group with $P \in \mathrm{Syl}_p\, G$. If $C_G(O_p(G)) \leqslant O_p(G)$, $p \neq 2$ and G does not involve $SL(2, p)$, then $K_\infty(P) \trianglelefteq G$.*

Proof. For parts (i) and (ii) see Proposition 12.1 of [8]. Part (iii) follows from Theorems 12.3(c) and 14.6 of [8].

Let G be a group containing a strongly closed 2-subgroup S and p be an odd prime. We define

$$И_G(S; p) = \{P \mid P\ p\text{-subgroup of } G,\ S \leqslant N_G(P) \text{ and } C_P(S) \leqslant O_{2'}(C_G(S))\},$$

and $И_G^*(S; p)$ to be the set of maximal elements of $И_G(S; p)$.

Our next result is in a similar vein to that of [10, (3.11)] and, likewise, will be applied in the "p-constrained situation" here.

LEMMA 3.9. *Let G be an $\mathscr{L}(S)$-group. Assume the following hold*:

(i) *$F^*(G)$ is a p-group, $p \neq 2$; and*

(ii) *$K = \langle S^G \rangle$ possesses no 2-components isomorphic to $SL(2, q)$ for any q a power of p, $q > 3$.*

Then for $P \in И_G^*(S; p)$ we have $K_\infty(P) \trianglelefteq G$.

Proof. Let $P \in И_G^*(S; p)$, and set $\bar{G} = G/O_{2'}(G)$. Note that $F^*(H)$ is a p-group whenever $H \geqslant F^*(G)$. It is an easy calculation (see [10, (3.10)(i)]) to show that $P \cap O_{2'}(G) \in \mathrm{Syl}_p\, O_{2'}(G)$. In particular, $P \geqslant O_p(G) = F^*(G)$. If $K_\infty(P) \leqslant O_{2'}(G)$ holds, then $K_\infty(P) = K_\infty(P \cap O_{2'}(G))$ by Lemma 3.8(ii). Then applying Lemma 3.8(iii) to $O_{2'}(G)$ we obtain $K_\infty(P) \leqslant O_p(G)$, and hence $K_\infty(P) = K_\infty(O_p(G)) \trianglelefteq G$. So we may suppose that $K_\infty(P) \nleqslant O_{2'}(G)$. Therefore, $[\overline{K_\infty(P)}, F^*(\bar{G})] \neq 1$.

Clearly $[\bar{P}, \bar{S}] \leqslant \bar{K}$, and so, using Lemma 3.6(ii),

$$[O_2(\bar{G}), [\bar{P}, \bar{S}]] \leqslant O_2(\bar{G}) \cap \bar{K} = O_2(\bar{K}) \leqslant S.$$

Hence, as $\bar{S} \leqslant N_{\bar{G}}(\bar{P})$, $[O_2(\bar{G}), [\bar{P}, \bar{S}]] = 1$. By the $P \times Q$ lemma $[O_{2'}(C_{\bar{G}}(\bar{S}), O_2(\bar{G})] = 1$ and so, as $C_{\bar{P}}(\bar{S}) \leqslant O_{2'}(C_{\bar{G}}(\bar{S}))$, we see that $\bar{P} = [\bar{P}, \bar{S}]\, C_{\bar{P}}(\bar{S})$ centralizes $O_2(\bar{G})$.

Observe that, by Lemma 3.1, every component of $\bar{G}$ is of type $\mathscr{L}$. Let $\bar{J}$ denote the product of all 2-components of $\bar{G}$ of type $L_2(2^n)$, $\mathrm{Sz}(2^n)$, $U_3(2^n)$,

$L_3(2^n)$ and $PSp_4(2^n)$. Since $\bar{S} \cap \bar{J}$ normalizes $\bar{P}$, $\bar{P} \cap \bar{J} = 1$ by (2.16)(iv) and Lemma 3.1, and so $[\bar{S} \cap \bar{J}, \bar{P}] = 1$, which then gives $[\bar{J}, \bar{P}] = 1$. Let L be the inverse image in G of all the components of $\bar{G}$ not in $\bar{J}$. So $[\overline{K_\infty(P)}, \bar{L}] \neq 1$. Set $H = LPS$. Note that we have $P \geqslant O_p(H)$. By hypothesis (ii) we see that Q_8 is not involved in LP. Suppose $SL(2, p)$ is involved in H. So there exists $H \geqslant A \geqslant B \geqslant C$ with $C \trianglelefteq A$ such that $A/C \cong SL(2, p)$ and $B/C = O^2(A/C)$. Let $P_0 \in \mathrm{Syl}_p A$. Then $[B, P_0]C/C$ contains a Sylow 2-subgroup of A/C. However, $P_0 \leqslant O^2(H) \leqslant LP$ forces $[B, P_0] \leqslant LP$ and then Q_8 is involved in LP. Thus $SL(2, p)$ is not involved in H. Let $Q \in \mathrm{Syl}_p H$ with $Q \geqslant P$. Employing Lemma 3.8(iii) yields $K_\infty(Q) \leqslant O_p(H) \leqslant P$ and hence $K_\infty(P) = K_\infty(Q)$. But then $[\overline{K_\infty(P)}, \bar{L}] \leqslant \overline{O_p(H)} \cap \bar{L} = 1$, contrary to $[\overline{K_\infty(P)}, \bar{L}] \neq 1$.
The proof of the lemma is complete.

4. The Minimal Counterexample

For the remainder of this paper G is a non-abelian simple group containing a non-trivial strongly closed 2-subgroup S with $\mathrm{cl}\, S \leqslant 2$ such that $G \notin \mathscr{L}$ and, subject to this, we minimize $|G| + |S|$. As mentioned in the Introduction, we will now investigate the internal structure of G with the aim of obtaining a contradiction. We begin our campaign with the following lemma.

Lemma 4.1. (i) *If* $1 \neq S_0 < S$, *then* S_0 *is not a strongly closed* 2-*subgroup of* G *and* $\mathrm{cl}\, S = 2$.

(ii) $Z(S)$ *is elementary abelian and* $\mho'(S) \leqslant Z(S)$.

(iii) $Z(S)$ *is non-cyclic and* $\mathrm{SCN}_3(S) \neq \varnothing$.

(iv) $G = \langle C_G(\sigma) \mid \sigma \in \mathscr{I}(S) \rangle$.

(v) *If* $H < G$ *and* $R \in \mathscr{X}^*(H)$, *then* H *is an* $\mathscr{L}(R)$-*group.*

(vi) *If* S *contains an abelian subgroup of index* 2, *then* $|S'| = 2$.

Proof. By (2.9) and the minimality of $|G| + |S|$, (i) is clear, and part (ii) follows from (2.8). If $Z(S)$ were cyclic, then, by (i) and (ii), S is extraspecial and hence (2.10) implies that $G \in \mathscr{L}$. Thus $Z(S)$ is non-cyclic. By part (i), $\Omega_1(S) \nleqslant Z(S)$ and so, since $\mathrm{cl}\, S = 2$, $\mathrm{SCN}_3(S) \neq \varnothing$. So part (iii) holds. From (2.4) and a result of Bender [2] we obtain (iv), while part (v) is obvious. By (ii) and (2.12), part (vi) holds.

Lemma 4.2. (i) $\mathscr{C}(S) \neq \varnothing$.

(ii) *If* $\sigma \in \mathscr{C}(S)$ *and* $H \geqslant C_G(\sigma)$, *then* $E_{2'}(H) \neq 1$.

Proof. (i) Suppose $\mathscr{C}(S) = \varnothing$. Thus $C_G(\sigma)$ is 2-constrained for all

$\sigma \in \mathscr{I}(S)$. By Lemma 4.1(iii) and standard signalizer functor arguments $\langle C_G(\sigma) \mid \sigma \in \mathscr{I}(S) \leqslant N_G(\langle O_{2'}(C_G(\sigma)) \mid \sigma \in \mathscr{I}(S) \rangle)$ whence, by Lemma 4.1(iv), $O_{2'}(C_G(\sigma)) = 1$ for all $\sigma \in \mathscr{I}(S)$. In other words $F^*(C_G(\sigma)) = O_2(C_G(\sigma))$ for all $\sigma \in \mathscr{I}(S)$ and hence (2.13)(v) gives $F^*(H) = O_2(H)$ for all $H = N_G(S_0)$, where $1 \neq S_0 \leqslant S$. But then $G \in \mathscr{L}$ by (2.11), a contradiction. Therefore, $\mathscr{C}(S) \neq \varnothing$.

(ii) This follows from [15, Theorem 3.1].

LEMMA 4.3. *For each $\sigma \in \mathscr{I}(S)$, $C_G(\sigma)$ has no 2-components isomorphic to $SL(2, q)$, q odd.*

Proof. Assume that the lemma is false and let $\sigma \in \mathscr{I}(S)$ be such that $C_G(\sigma)$ has a 2-component K with $K/O_{2'}(K) \cong SL(2, q), q$ odd. Let $R \in \mathscr{X}^*(C_G(\sigma))$. By (2.1) and Lemmas 3.1 and 3.3(i), $K \cap R \in \mathrm{Syl}_2 K$ and R normalizes K. Now imitating the proof of Proposition 4.1 of [5] yields that $Z(S)$ contains a non-trivial stronly closed 2-subgroup, contrary to Lemma 4.1(i).

5. UNIQUENESS THEOREMS

In this section we establish some analogues of results in Section 5 of [10]. The fact that Lemma 3.4 is not a strong as (3.7) of [10] leads to some extra complications.

Let $\sigma \in \mathscr{I}(S)$. As in [10] we define $\mathscr{M}(\sigma)$ to be $\mathscr{M}_1(\sigma)$ if $\mathscr{M}_1(\sigma) \neq \varnothing$ and $\mathscr{M}_2(\sigma)$ if $\mathscr{M}_1(\sigma) = \varnothing$ where $\mathscr{M}_1(\sigma) = \{H \mid H$ is a maximal subgroup of G containing $C_G(\sigma)$ and $O_p(H) \neq 1 = C_{O_p(H)}(\sigma)$ for some odd prime $p\}$; and $\mathscr{M}_2(\sigma) = \{H \mid H$ is a maximal subgroup of G containing $C_G(\sigma)$ with $|E(H)|$ maximal$\}$.

Also, for $H \in \mathscr{M}(\sigma)$, $\mathfrak{U}_\sigma(H)$ denotes the set of non-trivial $\langle \sigma \rangle$-invariant subnormal subgroups Y of $F^*(H)$ for which $N_G(Y) \leqslant M$, M a maximal subgroup of G, implies either that $M = H$ or that $F^*(H)$ and $F^*(M)$ are p-groups for the same prime p. When there is no danger of confusion we shall just write $\mathfrak{U}_\sigma(H)$ as $\mathfrak{U}(H)$.

For the remainder of this section, $\sigma \in \mathscr{I}(S)$, $H \in \mathscr{M}(\sigma)$ and $R \in \mathscr{X}^*(H)$ with $S \cap H \leqslant R$.

LEMMA 5.1. *If $1 \neq Y \trianglelefteq \trianglelefteq F^*(H)$ and Y is $C_G(\sigma)$-invariant, then $Y \in \mathfrak{U}(H)$.*

Proof. The same proof as for [10, (5.2)], using Lemma 3.4 in place of [10, (3.7)], establishes the result.

The next lemma is a modified form of [10, (7.3)]. But first we must introduce some additional notation.

For $\sigma_0 \in \mathscr{I}(S)$ and K a quasisimple group we define $\mathscr{H}(\sigma_0, K) = \{H_0 \in \mathscr{M}(\sigma_0) \mid \text{if } K_0 = K_0^* \leqslant H_0 \text{ with } K_0 \cong K, \text{ then } [K_0, O_{2'}(H_0)] = 1\}$.

LEMMA 5.2. *Let A be a maximal elementary abelian subgroup of S. Suppose $\sigma \in A^{\#}$, $H \in \mathscr{M}(\sigma)$ and K is a component of $\langle R^H \rangle$ with σ, H and K chosen so that, if possible, K is not of type $L_2(q)$ ($q \equiv 3, 5(8)$, $q > 3$) or A_6. If $A \notin \mathfrak{A}_e(S)$, we further assume that*

(i) *for all $\mu \in A^{\#}$ and $H_\mu \in \mathscr{M}(\mu)$, A normalizes each component of H_μ; and*

(ii) *if it is the case that $K \cong L_2(2^n)$, some n, then choose K so as n is as large as possible.*

If there exists $\sigma_1 \in (A \cap K)^{\#}$ such that $\mathscr{H}(\sigma_1, K) \neq \varnothing$, then $\mathscr{H}(\sigma_0, K) \subseteq \{H\}$ for each $\sigma_0 \in C_A(K)^{\#}$.

Proof. We begin by showing that $E(H) = E(KC_G(K))$. Let E_0 be the product of all components of $E(H^*)$ which are isomorphic to K. Clearly $E_0 \trianglelefteq H$. If $K = E_0$, then $KC_G(K) \trianglelefteq H$ and so $E(H) = E(KC_G(K))$. Therefore, we may suppose that $E_1 = E(C_{E_0}(K)) \neq 1$. By hypothesis there exists $\sigma_1 \in (A \cap K)^{\#}$ and $H_1 \in \mathscr{H}(\sigma_1, K)$. Note that $\sigma \in H_1$. From Lemma 3.1, $E_1 = E_1^*$ and hence, since $E_1 \leqslant H_1$, $[E_1, O_{2'}(H_1)] = 1$. Now E_1 is a product of components of $E(H_1 \cap E_0)$ and $E(H_1 \cap E_0)$ is $C_{H_1}(\sigma)$-invariant. Hence, by Lemmas 3.2(ii) and 3.4(ii) and the choice of σ, H and K, either E_1 is a product of components of $E(H_1)$ or there exists a component K_1 of E_1 and a component L_1 of H_1 such that $K_1 < L_1$ with $K_1 \cong L_2(2^n)$ and $L_1 \cong L_2(2^{2n})$ ($n \geqslant 2$) and $A \in \mathfrak{A}_e(S)$. Moreover, in the latter case, σ must induce a field automorphism upon L_1. However, by (2.17) and Lemma 3.3(ii), $A = C_A(L_1)T_1$ for some $T_1 \in \mathrm{Syl}_2 L_1$, and so the first alternative must hold. Therefore, as $C_G(K) \leqslant H_1$, E_1 is a product of components of $E(C_G(K))$.

Because $E(KC_G(K)) = KE(C_G(K))$ and $E_0 = KE_1$, we have that E_0 is a product of components of $E(KC_G(K))$, and so $E(KC_G(K)) \leqslant N_G(E_0) = H$. Let L be a component of $E(KC_G(K))$. Note that $F^*(H) \leqslant KC_G(K)$. Hence, if $L \nleqslant E(H)$, then $[L, E(H)] = 1$ by (2.13). Since $F(H)$ is a soluble subgroup of $KC_G(K)$ normalized by L, we obtain $L \leqslant C_H(F^*(H)) \leqslant F^*(H)$, a contradiction. Therefore, $E(KC_G(K))$ is a product of components of $E(H)$ containing E_0. Thus, by Lemma 3.1(i), we see that E_0 is the product of all components of $(KC_G(K))^*$ isomorphic to K. So $E_0 \trianglelefteq KC_G(K)$ and therefore $KC_G(K) \leqslant H$, which yields the desired $E(KC_G(K)) = E(H)$.

Let $\sigma_0 \in C_A(K)^{\#}$ and $H_0 \in \mathscr{H}(\sigma_0, K)$. By Lemma 3.1(i), $K = K^* \leqslant C_G(\sigma_0) \leqslant H_0$. Hence $[K, O_{2'}(H_0)] = 1$. Since K is a component of $E(E(H) \cap H_0)$, which is $C_{H_0}(\sigma)$-invariant, Lemmas 3.2(ii) and 3.4(ii) together with the choice of σ, H and K imply that K is a component of

$E(H_0)$. Repeating the above argument with H_0 in place of H yields that $E(KC_G(K)) = E(H_0)$. Consequently $H_0 = N_G(E(KC_G(K))) = H$. Thus $\mathscr{H}(\sigma_0, K) \subseteq \{H\}$ for each $\sigma_0 \in C_A(K)^{\#}$.

We note that the proof of Lemma 5.2 also establishes

LEMMA 5.3. *Let K be a component of $\langle R^H \rangle$ for which $R \cap K$ is non-abelian and $K \not\cong A_6$. Further assume there exists $\sigma_1 \in \mathscr{I}(K)$ such that $|\sigma, \sigma_1| = 1$ and $\mathscr{H}(\sigma_1, K) \neq \varnothing$. Then $\mathscr{H}(\sigma_0, K) \subseteq \{H\}$ for each $\sigma_0 \in C_{C_S(\sigma)}(K)^{\#}$.*

LEMMA 5.4. *Suppose that Y and V are non-trivial $\langle \sigma \rangle$-invariant subnormal subgroups of $F^*(H)$ with V normalizing Y. Also assume that either*

(a) *$V \in \mathfrak{U}(H)$ and $V = [V, \sigma]$ contains no components isomorphic to A_6; or*

(b) *$V \in \mathfrak{U}(H)$ and $V = [W, \sigma]$ where W is the product of all components of $E(H)$ isomorphic to A_6; or*

(c) *$[V, \sigma] = 1$, V is elementary abelian of order p^2; p an odd prime and $\langle v \rangle \in \mathfrak{U}(H)$ for all $v \in V^{\#}$.*

Then $Y \in \mathfrak{U}(H)$.

Proof. (a) Let M be a maximal subgroup of G containing $N_G(Y)$, and set $X = M \cap F^*(H)$. Observe that $VE(H) \leqslant N_{F^*(H)}(Y) \leqslant X$ and that $X = F^*(X)$. Put $X_1 = UO_{2'}(X)$, where U is the product of all components of X not isomorphic to A_6. Then $V \trianglelefteq \trianglelefteq X_1$ and, clearly, X_1 is $C_M(\sigma)$-invariant. Appealing to Lemma 3.4(i) yields that $V \trianglelefteq \trianglelefteq [X_1, \sigma] \trianglelefteq \trianglelefteq F^*(M)$. The conclusion that $Y \in \mathfrak{U}(H)$ may now be obtained as in [10, (5.3)]. Using Lemma 3.4(ii) where appropriate in [10, (5.3)] also yields (c).

Now we prove (b). Suppose $N_G(Y) \leqslant M$, where M is a maximal subgroup of G. Since V is non-trivial subgroup of $F^*(H)$, we have $|\pi(F^*(H))| \geqslant 2$. Note that this implies $N_G(V) \leqslant H$. Arguing by way of a contradiction, we shall suppose $Y \notin \mathfrak{U}(H)$. Thus $H \neq M$.

Since $V \leqslant N_G(Y) \leqslant M$ and V is $C_G(\sigma)$-invariant, Lemma 3.4(i) yields that $V = V_1 V_2$, where $V_1 = K_1 \cdots K_s \trianglelefteq \trianglelefteq F^*(M)$ and $V_2 = K_{s+1} \cdots K_n$, where $K_i < L_i \trianglelefteq \trianglelefteq F^*(M)$ $(s+1 \leqslant i \leqslant n)$ with $L_i \cong A_7$ $(s+1 \leqslant i \leqslant n)$. (Here $K_i \cong A_6$ for $1 \leqslant i \leqslant n$.) Since $N_{F^*(H)}(Y) \leqslant M$ and $|\pi(F^*(H))| \geqslant 2$, (2.14) implies that $F^*(M) \nleqslant H$. Thus $V_2 \neq 1$ as $N_G(V) \leqslant H$.

Let $\tilde{V}$ be the product of all components of $E(H)$ not contained in V. Since $F(M) \leqslant N_G(V) \leqslant H$ and $E(H) \leqslant M$, $[E(H), F(M)] = 1$, and so $[E(H), O_{2'}(M)] = 1$. Because V is $C_G(\sigma)$-invariant $\tilde{V}$ is $C_G(\sigma)$-invariant. Thus $\tilde{V} \leqslant E(M)$ by Lemma 3.4(ii). Observe that $[\tilde{V}, L_{s+1} \cdots L_n] = 1$. If $\tilde{V} \neq 1$, then, by Lemma 5.1, $\tilde{V} \in \mathfrak{U}(H)$, and as a consequence $L_{s+1} \cdots L_n \leqslant$

$N_G(\tilde{V}) \leqslant H$. This gives the untenable $V_2 \trianglelefteq L_{s+1} \cdots L_n$. Therefore we have shown that

(5.1) $E(H) = V = V_1 V_2 \leqslant E(M)$.

Next we prove that

(5.2) $F^*(H) = E(H) \leqslant H^*$.

By Lemma 3.1(ii), $E(H) \leqslant H^*$.

Set $Z = N_{F(H)}(Y)$. Then, since $Z \leqslant M$ and $[Z, V] = 1$, Z normalizes each L_i and hence must centralise each L_i $(s+1 \leqslant i \leqslant n)$. Therefore, $F(H) \neq 1$ would force $L_{s+1} \cdots L_n \leqslant N_G(Z(F(H))) = H$, which is impossible. So $F(H) = 1$, and this proves (5.2).

(5.3) (i) $F^*(M) = E(M) = V_1 L_{s+1} \cdots L_n \leqslant M^*$.
(ii) *$M^*/E(M)$ is an elementary abelian 2-group.*

Let E be the product of all components of $E(M)$ not contained in $V_1 L_{s+1} \cdots L_n$. Then by (5.1) and (5.2), $EF(M) \leqslant C_G(E(H)) = C_H(E(H)) = 1$. Therefore, $F^*(M) = E(M) = V_1 L_{s+1} \cdots L_n$. From (5.1) we also deduce that $E(M) \leqslant M^*$ so establishing part (i).

If K is a component of $E(M)$, then $K \trianglelefteq M^*$ by Lemma 3.3(i) and so, as $\operatorname{Out} K$ is an elementary abelian 2-group, $M^*/E(M)$ is an elementary abelian 2-group.

(5.4) *We may assume that Y consists of one component of $E(H)$.*

By (5.2), $Y = \tilde{K}_1 \cdots \tilde{K}_t$, where the $\tilde{K}_i$ are components of $E(H)$. Subject to $1 \neq Y = Y^\sigma \trianglelefteq \trianglelefteq E(H)$ and $Y \notin \mathfrak{A}(H)$, choose Y so that t is minimal. Suppose $t > 1$. Then, since $\tilde{K}_i^\sigma = \tilde{K}_i$ by Lemma 3.2(i), $\tilde{K}_i \in \mathfrak{A}(H)$ for all i, $1 \leqslant i \leqslant t$. Hence, as $|\pi(F^*(H))| \geqslant 2$, this gives $E(M) \leqslant H$, a contradiction. Thus (5.4) holds.

(5.5) *For each l where $s+1 \leqslant l \leqslant n$ and j where $1 \leqslant j \leqslant s$, there exists $\tilde{h} \in H$ such that $K_j^{\tilde{h}} = K_l$.*

Suppose (5.5) were false, and set $K = \langle K_l^h \mid h \in H \rangle$. Then $[K, L_l] = 1$ and hence, since K is $C_G(\sigma)$-invariant, $L_l \leqslant N_G(K) \leqslant H$, a contradiction. This proves (5.5).

Recall that $R \in \mathscr{X}^*(H)$ with $\sigma \in R$. From (2.1)(ii) and (5.3)(i) for $1 \leqslant i \leqslant n$, $R \cap K_i$ is a non-trivial strongly closed 2-subgroup of K_i, and thus $R \cap K_i \in \operatorname{Syl}_2 K_i$.

(5.6) $[\sigma, R^h \cap E(H)] = 1$ *for some $h \in E(H)$.*

Since $K_i \trianglelefteq H^*$ $(1 \leqslant i \leqslant n)$, either $\sigma \in C_{\langle\sigma\rangle K_i}(K_i)K_i$ or $\langle\sigma\rangle K_i \cong S_6$. In either case $[\sigma, R^{k_i} \cap K_i] = 1$ for some $k_i \in K_i$, and so $[\sigma, R^h \cap E(H)] = 1$, where $h = k_1 \cdots k_n$.

Without loss of generality, we will suppose $h = 1$ in (5.6).

(5.7) $\mathscr{H}(\tau, A_7) = \mathscr{M}(\tau)$ *for each* $\tau \in \mathscr{I}(R \cap K_i)$, *each* i, $1 \leqslant i \leqslant n$.

Let $\tau \in \mathscr{I}(R \cap K_i)$, and suppose for the moment that $i \neq n$. By (5.4) and (5.5) there exists $\tilde{h} \in H$ such that $Y^{\tilde{h}} = K_n < L_n$. Clearly $N_G(K_n) = N_G(Y^{\tilde{h}}) < M^{\tilde{h}} = \tilde{M}$. Let $H_\tau \in \mathscr{M}(\tau)$. In order to prove that $H_\tau \in \mathscr{H}(\tau, A_7)$ we must show that $L = L^* \leqslant H_\tau$ with $L \cong A_7$ implies that $[L, O_{2'}(H_\tau)] = 1$. Since $i \neq n$, $L_n \leqslant C_G(\tau) \leqslant H_\tau$ and, by (5.6), $\sigma \in H_\tau$. Now $K_n \trianglelefteq \trianglelefteq E(E(H) \cap H_\tau)$ and $E(E(H) \cap H_\tau)$ is $C_{H_\tau}(\sigma)$-invariant. Therefore, by Lemma 3.4(i), $K_n \leqslant \hat{J}$, where $\hat{J}$ is a component of H_τ with $\hat{J} \cong A_6$ or A_7. Since $1 \neq K_n \leqslant \hat{J} \cap L_n \trianglelefteq \trianglelefteq L_n$, we see that $L_n \leqslant E(H_\tau)$ and hence that $L_n = \hat{J}$. Thus $L_n = L_n^* \trianglelefteq H_\tau^*$ by Lemma 3.3. Because $[H_\tau^* : N_{H_\tau^*}(K_n)] = 7$, $[L : L_0] = 1$ or 7 where $L_0 = L \cap N_{H_\tau^*}(K_n)$. Therefore, L_0 is isomorphic to either A_6 or A_7 and $L_0^* = L_0$. Since $N_G(K_n) \leqslant \tilde{M}$, $O_{2'}(H_\tau) \leqslant \tilde{M}$ and, as $\tilde{M}$ is conjugate to M, $L_0 \leqslant E(\tilde{M})$ by (5.3)(ii). From the structure of $E(\tilde{M})$ we see that $[L_0, [L_0, O_{2'}(H_\tau)]] = 1$. Hence $[L_0, O_{2'}(H_\tau)] = 1$ and so, as L is simple, $[L, O_{2'}(H_\tau)] = 1$. Thus $H_\tau \in \mathscr{H}(\tau, A_7)$, and we have proved (5.7) when $i \neq n$. Since $n > 1$, the case $i = n$ follows using (5.5).

(5.8) *Let* $1 \leqslant j < n$. *Then* $\{M\} = \mathscr{M}(\rho)$ *for each* $\rho \in \mathscr{I}(K_j)$.

Let $1 \neq \tau \in Z(R \cap K_j)$ and let $N \in \mathscr{M}(\tau)$. Clearly $L_n \leqslant N$ and, by (5.6), $\sigma \in N$. Arguing as in (5.7) using Lemma 3.4(i) we obtain $L_n \trianglelefteq \trianglelefteq E(N)$. By (5.7) there exists $\tau_1 \in \mathscr{I}(L_n)$ with $[\tau, \tau_1] = 1$ and $\mathscr{H}(\tau_1, A_7) \neq \varnothing$. Lemma 5.3 (with $N = H$, $\tau = \sigma$ and $L_n = K$) and (5.7) then yield that $\mathscr{M}(\zeta) = \{N\}$ for all $\zeta \in \mathscr{I}(R \cap K_j)$. We now show that $N = M$. Now $K_j \leqslant \hat{L}$, where $\hat{L}$ is a component of $E(M)$ isomorphic to either A_6 or A_7 and consequently

$$E(M) \leqslant \langle C_G(\zeta) \mid \zeta \in \mathscr{I}(R \cap K_j)\rangle \leqslant N.$$

Since σ, $V \leqslant N$ and V in $C_N(\sigma)$-invariant, $V \leqslant E(N)$ by Lemma 3.4(i). Hence, as $1 \neq V \leqslant E(N) \cap E(M) \trianglelefteq \trianglelefteq E(M)$, we obtain $E(M) \leqslant E(N)$. Appealing to Lemma 3.4(i) as above also gives $L_{s+1} \cdots L_n \trianglelefteq \trianglelefteq E(N)$. Since $L_{s+1} \cdots L_n \trianglelefteq M$ and, by Lemma 3.1, $E(N) = E(N)^*$, (5.3)(ii) implies that $E(N) \leqslant E(M)$. Hence $E(M) = E(N)$ and therefore $M = N$. So $\{M\} = \mathscr{M}(\zeta)$ for all $\zeta \in \mathscr{I}(R \cap K_j)$ and hence, as K_j has only one conjugacy class of involutions, $\{M\} = \mathscr{M}(\rho)$ for each $\rho \in \mathscr{I}(K_j)$.

We are now in a position to obtain the desired contradiction. Let $g \in C_G(\sigma)$. Then, as $C_G(\sigma) \leqslant H$, $V \leqslant M \cap M^g$. Since $\sigma \in M$, we also have $\sigma \in M^g$. Employing Lemma 3.4(i) on M^g gives that $V \leqslant E(M^g)$ and that K_1 is contained in a component of M^g isomorphic to either A_6 or A_7. Therefore, using (5.8), we have

$$E(M^g) \leqslant \langle C_G(\zeta) \mid \zeta \in \mathscr{I}(K_1) \rangle \leqslant M,$$

whence $E(M^g) \leqslant E(M)$ by (5.3). Thus $g \in N_G(E(M)) = M$, and so $C_G(\sigma) \leqslant M$. But then this in contrary to the definition of $\mathscr{M}(\sigma)$ since $|E(H)| < |E(M)|$. With this contradiction the proof of part (b) is complete, and we have established Lemma 5.4.

For $H \in \mathscr{M}(\sigma)$, we set $F_0^*(H) = F(H)' \, C_{F^*(H)}(F(H)')$. Clearly $F_0^*(H)$ is a characteristic subgroup of $F^*(H)$, and $F_0^*(H) = E(H) \, P_1 \cdots P_r$ where $P_i = O_{p_i}(H)' \, C_{O_{p_i}(H)}(O_{p_i}(H)')$, $p_i \in \pi(F(H))$.

Lemma 5.5. *Let p be an odd prime, and set $P = O_p(H)$. Suppose that $[P, \sigma] \neq 1$, $Cp(\sigma) \leqslant Z(P)$ and $Z(P)$ is cyclic. If there exists $\mu \in \sigma^G \cap C_G(\sigma)$ such that $[Cp(\mu), \sigma] \neq 1$, then $F^*(H)$ is a p-group.*

Proof. Set $\mu = \sigma^g$. Then $H^g \in \mathscr{M}(\mu)$. Since $[Cp(\mu), \sigma] \neq 1$, there exists $Y \leqslant Cp(\mu)$ of order p and inverted by σ. Now $[O_p(H) \cap H^g, \sigma]$ is $C_{H^g}(\sigma)$-invariant whence $Y \leqslant O_p(H^g) = P^g$ by Lemma 3.4(i). Hence $C_{P^g}(\mu) \neq 1$ and so, as $Z(P)$ is cyclic, $C_{P^g}(\mu) = Z(P^g)$. So $C_P(\sigma) = Z(P)$ and therefore $g \notin H$. Thus, since $Y = \Omega_1(Z(P^g))$, $N_G(Y) = H^g \neq H$.

Because $[P, \sigma] \neq 1$, $P \neq Z(P)$ and so $C_{P^g}(\sigma) \neq 1$. Since σ does not centralize $Y = \Omega_1(C_{P^g}(\mu))$, $[C_{P^g}(\sigma), \mu] \neq 1$. So we may choose a subgroup $Y_1 \leqslant C_{P^g}(\sigma)$ of order p and inverted by μ, and by repeating the above argument obtain $H = N_G(Y_1)$. Then $F^*(H)$ is a p-group by (2.14).

Lemma 5.6. *Suppose $[O_p(H), \sigma] \neq 1$, where p is some odd prime. Then either*

(i) *$Y \in \mathfrak{A}(H)$ for each $1 \neq Y = Y^\sigma \trianglelefteq \trianglelefteq F_0^*(H)$; or*

(ii) *$R \cap O_2(H) \neq 1$.*

Moreover, if (i) *does not hold, then there exists $\mu \in \mathscr{I}(O_2(H) \cap R)$ such that $H \in \mathscr{M}(\mu)$.*

Proof. Set $P = O_p(H)$, and $P_0 = O_p(F_0^*(H)) = P \cap F_0^*(H)$. We suppose (i) does not hold. So there exists $1 \neq Y = Y^\sigma \trianglelefteq \trianglelefteq F_0^*(H)$ such that $Y \notin \mathfrak{A}(H)$.

Since $C_P(P_0) \leqslant P_0$, we have $[P_0, \sigma] \neq 1$ and hence, because $N_{P_0}(Y) = N_{P_0}(Y \cap P_0)$, $[N_{P_0}(Y), \sigma] \neq 1$. So there exists $Y_1 \leqslant N_{P_0}(Y)$ with $|Y_1| = p$ and Y_1 inverted by σ. *If* $Y_1 \in \mathfrak{A}(H)$, then Lemma 5.4(a) forces $Y \in \mathfrak{A}(H)$. So $Y_1 \notin \mathfrak{A}(H)$ and we may take $Y = Y_1 \leqslant P_0$.

Let Q be a $C_G(\sigma)$-invariant subgroup of P containing Y. Then Lemmas 5.1 and 5.4(a) imply $[C_P(Q), \sigma] = 1$. Since $1 \neq Y \leqslant [Q, \sigma] \in \mathfrak{A}(H)$ by Lemma 5.1, $\langle x \rangle \in \mathfrak{A}(H)$ for all $x \in C_P(Q)^{\#}$. Therefore, $C_P(Q)$ is cyclic by Lemma 5.4(c).

With $Q = P_0$ we obtain $Z(P_0) \leqslant C_P(\sigma)$ and $Z(P_0)$ cyclic whence $Z(P')$ is cyclic and, by [18, p. 303], P' is cyclic. Noting that $P' \leqslant C_P(\sigma)$, we have $C_P(\sigma) \trianglelefteq P$, which gives $[C_P(\sigma), [P, \sigma]] = 1$. Since $P = C_P(\sigma)[P, \sigma]$, this gives $Z(P) \leqslant Z(P_0) \leqslant C_P(\sigma) \leqslant Z(P)$, and so $Z(P) = C_P(\sigma)$.

Let $P_1 = [P, \sigma]$. Since $P_1' \leqslant P' \leqslant C_P(\sigma)$, P_1 has class at most two by the three subgroups lemma. Let $P_2 = \Omega_1(P_1)$. Noting that $Y \leqslant P_2$ we deduce that $Z(P_2)$ is cyclic. Hence, by [12, Lemma 5.3.9], P_2 is extra-special. Let $Y_0 = [C_{P_2}(Y), \sigma]$. Since $[P_1, \sigma] \neq 1$, $[P_2, \sigma] \neq 1$ and hence $Y_0 \neq 1$. From Lemma 5.4(a) $Y_0 \notin \mathfrak{A}(H)$. We claim that Y_0 is abelian. Suppose this were false. Then $Y_0' = Z(P_2)$ and so $N_G(Y_0) \leqslant N_G(Z(P_2)) = H$. Suppose $N_G(Y_0) \leqslant M$, where M is any maximal subgroup of G. Then $N_G(Y_0) \leqslant H \cap M$ and $Y_0 \leqslant P \cap M$. Therefore, $Y_0 \leqslant [P \cap M, \sigma] \leqslant O_p(M)$ by Lemma 3.4(i) and so $Y_0 \in \mathfrak{A}(H)$ by (2.14), a contradiction. Thus Y_0 is abelian, as claimed. Consequently $C_{P_2}(Y) = Y_0(C_{P_2}(Y) \cap C_P(\sigma)) \leqslant Y_0 Z(P)$ is abelian, which, by [18, p. 353], forces $|P_2| = p^3$.

(5.9) *$F^*(H)$ is not a p-group.*

Let $\xi \in R$. If $[P_2, \xi] = 1$, then $[P_1, \xi] = 1$. Since ξ centralizes $Z(P) = \Omega_1(C_P(\sigma))$, we see that $[P, \xi] = 1$.

Now suppose $F^*(H)$ is a p-group. Then R must act faithfully upon P_2 which, as $\operatorname{Out} P_2 \cong GL(2, p)$, gives $m(R) \leqslant 2$. However, by Lemma 4.1(i), (iii), $m(R) > 3$. Therefore, $F^*(H)$ is not a p-group.

(5.10) *Let $\sigma^g \in C_G(\sigma)$ with $\sigma^g \neq \sigma$. Then* $[P, \sigma\sigma^g] = 1$.

Recall that $C_P(\sigma) = Z(P)$ is cyclic and hence, by Lemma 5.5 and (5.9), $C_P(\sigma^g) \leqslant C_P(\sigma)$. Since $P \neq Z(P)$, $C_P(\sigma^g) \neq 1$ and so $C_P(\sigma^g) = C_P(\sigma)$. Therefore, $[P, \sigma\sigma^g] = 1$ by [12, Theorem 6.2.4].

(5.11) (i) *$P_1 \in \mathfrak{A}_\sigma(H)$ and $N_G(P_1) \leqslant H$.*

(ii) *$[O^{\{2,p\}}(F^*(H)), \sigma] = 1$ and $O_q(H)$ is cyclic for all $q \in \pi(F(H)) \backslash \{p, 2\}$.*

From Lemma 5.1, $P_1 = [P, \sigma] \in \mathfrak{A}_\sigma(H)$, and so $N_G(P_1) \leqslant H$ by (5.9). If $[O^{\{2,p\}}(F^*(H)), \sigma] \neq 1$, then applying one of Lemma 5.4(a), (b) we obtain $Y \in \mathfrak{A}_\sigma(H)$, a contradiction. Therefore, $[O^{\{2,p\}}(F^*(H)), \sigma] = 1$. Let $q \in \pi(F(H)) \backslash \{2, p\}$. Then $\langle x \rangle \in \mathfrak{A}_\sigma(H)$ for all $x \in O_q(H)^{\#}$ by (i) and

Lemma 5.4(a). Hence, as $Y \notin \mathfrak{U}_\sigma(H)$, $O_q(H)$ is cyclic by Lemma 5.4(c). This proves (ii).

Let $N_G(Y) \leqslant M$, where M is a maximal subgroup of G. Since $Y \notin \mathfrak{U}_\sigma(H)$ and $F^*(H)$ is not a p-group, $H \neq M$.

(5.12) *Let $\xi \in \mathscr{I}(C_H(P) \cap C_G(\sigma))$ and let $N \in \mathscr{M}(\xi)$. Then*

(i) *$P_1 = [P, \sigma] \leqslant O_p(N)$ and (hence) $O_p(N)$ is non-abelian;*

(ii) *$\tilde{R} \cap O_2(N) \neq 1$, where $\tilde{R} \in \mathscr{X}^*(N)$;*

(iii) *if $O_p(N) \leqslant H$, then $H = N$;*

(iv) *if $N \neq H$, then $O_q(H) \cap N = 1$ for all $q \in \pi(F(H)) \backslash \{2, p\}$; and*

(v) *$O_r(M) \cap N = 1$ for all $r \in \pi(F(M)) \backslash \{2, p\}$.*

(i) By hypothesis $\sigma, P \leqslant C_G(\xi) \leqslant N$ and so Lemma 3.4(i) gives $P_1 = [P, \sigma] \leqslant O_p(N)$. The second assertion of (i) also follows since P_1 is non-abelian.

(ii) Since $O_p(N) \cap H$ is $C_H(\xi)$-invariant, Lemma 3.4(i) yields $[O_p(N) \cap H, \xi] \leqslant P$. Hence, since $[P, \xi] = 1$, $[O_p(N) \cap H, \xi] = 1$ and consequently, as $N_{O_p(N)}(P_1) \leqslant O_p(N) \cap H$, $[O_p(N), \xi] = 1$ by the $P \times Q$ lemma.

Set $X = O^{\{2,p\}}(F^*(N))$, and suppose that $[X, \xi] \neq 1$. By Lemma 5.1, $[X, \xi] \in \mathfrak{U}_\xi(N)$ and then $P_1 \in \mathfrak{U}_\xi(N)$ by Lemmas 5.4(a), (b). Since $N_G(P_1) \leqslant H$ and $P_1 \leqslant O_p(N)$, (5.9) and (2.14) force $H = N$. Now ξ and σ commute and so $[X, \xi]$ is σ-invariant. Thus $[X, \xi] \in \mathfrak{U}_\sigma(H)$. Then $Y \in \mathfrak{U}_\sigma(H)$ by Lemmas 5.4(a), (b), a contradiction. Therefore, $[X, \xi] = 1$, and so $[\xi, O^2(F^*(N))] = 1$. Now (ii) follows using Lemma 3.2.

(iii) Suppose $O_p(N) \leqslant H$. From $N_G(P_1) \leqslant H$ (see (5.11)(i)) and part (i), (2.14) gives $[O_2(H), O_p(N)] = 1$. Thus $O_2(H) \leqslant N$. Let $q \in \pi(F(H)) \backslash \{2, p\}$. Then $[O_q(H), O_p(N)'] = 1$ by (5.11)(ii). Clearly $O_p(N)$ is a $C_H(\xi)$-invariant subgroup of H and so $[E(H), O_p(N)'] = 1$ by Lemma 3.4(iii). Since $O_p(N)$ is non-abelian and we already have $O_2(H)\, O_p(H) \leqslant N$, we obtain $F^*(H) \leqslant N$. Together with $N_{F^*(N)}(P_1) \leqslant H$, (5.9) and (2.14) this gives $H = N$.

(iv) Clearly $[O_q(H) \cap N, O_p(N) \cap H] = 1$. Then, since $N_G(P_1) \leqslant H$, $[O_q(H) \cap N,\ O_p(N)] = 1$ by the $P \times Q$ lemma. From (5.11)(ii) we have $O_q(H) \cap N \trianglelefteq H$ and therefore $O_q(H) \cap N \neq 1$ implies that $O_p(N) \leqslant H$. By (iii), $H = N$, against the supposition $H \neq N$. Hence $O_q(H) \cap N = 1$.

(v) By (i) we have $Y \leqslant P_1 \leqslant O_p(N)$. Also note that $N_{F^*(N)}(Y) \leqslant M$. Let $r \in \pi(F(M)) \backslash \{2, p\}$. If $r \notin \pi(F(N))$, then $O_r(M) \cap N = 1$ by (2.14). So we may suppose $r \in \pi(F(N))$. Hence $[O_r(M), O^r(F^*(N))] = 1$ by (2.14). So $[O_r(M), O_p(N)] = 1$. Thus $O_r(M) \leqslant N_G(P_1) \leqslant H$. Because $N_{F^*(H)}(Y) \leqslant M$, using (2.14) again gives that $r \in \pi(F(H))$. Thus $[O_r(M), O^r(F^*(H))] = 1$ by

(2.14). Consequnetly $P_1 \leqslant O_p(H) \leqslant O^r(F^*(H)) \leqslant M$. Since P_1 is $C_M(\sigma)$-invariant, Lemma 3.4(i) implies that $P_1 \leqslant O_p(M)$. Now $N_{F^*(M)}(P_1) \leqslant H$, $N_{F^*(H)}(Y) \leqslant M$ and (5.9) force $H = M$, which is not the case. With this contradiction we have (v).

By Glauberman's Z^*-theorem there exists $\mu \in \sigma^G \cap C_G(\sigma)$ with $\mu \neq \sigma$. Set $\rho = \mu\sigma$, and let $K \in \mathscr{M}(\rho)$. Without loss of generality, $\rho \in R$.

(5.13) (i) *We may suppose* $H \neq K$.

(ii) $O_p(K) \nleqslant H$.

(iii) $\sigma^H \cap C_G(\sigma) = \{\sigma\}$, *and so* $\sigma \in Z(R)$.

By (5.10), $[P, \rho] = 1$, and so $\rho \in C_H(\rho) \cap C_G(\sigma)$. If $H = K$, then $R \cap O_2(H) \neq 1$ by (5.12)(ii), whence the lemma is proven. So (i) holds. Then (ii) follows by using (5.12)(iii).

Suppose there exists $h \in H$ such that $\sigma^h \neq \sigma$ and, $\sigma^h \in C_G(\sigma)$. Because $Y \notin \mathfrak{U}_\sigma(H)$, $Y^h \notin \mathfrak{U}_{\sigma^h}(H)$. Since $H \in \mathscr{M}(\sigma^h)$, $[O^{\{2,p\}}(F^*(H)), \sigma^h] = 1$ by Lemmas 5.1 and 5.4(a), (b). Appealing to (5.10) and (5.11)(ii) gives $[O^2(F^*(H)), \sigma\sigma^h] = 1$. So $R \cap O_2(H) \neq 1$ by Lemma 3.3, proving the lemma. Hence $\sigma^H \cap C_G(\sigma) = \{\sigma\}$ must hold.

(5.14) *K does not contain any components of* $E(H)$.

Suppose (5.14) is false. Then $X = E(E(H) \cap K) \neq 1$. So $[X, N_{F^*(K)}(P_1)] \leqslant E(H) \cap F^*(K)$ which, by (2.13), yields that $1 \neq [X, E(K)] \trianglelefteq \trianglelefteq E(K)$ with $[X, E(K)]$ a product of components of $E(H)$. Because $P_1 = [P_1, \sigma] \in \mathfrak{U}_\sigma(H)$ and $[X, E(K)]$ is σ-invariant by (5.11)(ii), Lemma 5.4(a) implies that $[X, E(K)] \in \mathfrak{U}_\sigma(H)$. So $N_G([X, E(K)]) \leqslant H$ by (5.9). But then $O_p(K) \leqslant H$, contradicting (5.13)(ii). This establishes (5.14).

(5.15) (i) $[E(H), \rho] = E(H) \neq 1$.

(ii) $[E(H), O_{2'}(M)] = 1$.

(i) Combining (2.13) and (5.14) gives $[E(H), \rho] = E(H)$. We now prove that $E(H) \neq 1$. Suppose $E(H) = 1$ and argue for a contradiction. Since $[O_2(H), O_p(K)] = 1$ and, by (5.13)(ii), $O_p(K) \nleqslant H$, we have $O_2(H) = 1$. So, by (5.9), $O_{p'}(F(H))$ is a non-trivial group of odd order. Let $\xi \in \mathscr{I}(C_R(P))$ and let $N \in \mathscr{M}(\xi)$. By (5.13)(iii), $\xi \in C_G(\sigma)$. If $H = N$, then $R \cap O_2(H) \neq 1$, which yields the lemma. Therefore, we may suppose that for each $\xi \in \mathscr{I}(C_R(P))$, $H \neq N$, where $N \in \mathscr{M}(\xi)$. Therefore, by (5.12)(iv), ξ inverts $O_{p'}(F(H))$ for each $\xi \in \mathscr{I}(C_R(P)$ and hence $|\mathscr{I}(C_R(P))| = 1$.

Suppose $\sigma^x, \sigma^y \in C_R(\sigma)$ with $\sigma^x \neq \sigma \neq \sigma^y$. By (5.10), $\sigma\sigma^x, \sigma\sigma^y \in C_R(P)$ and thus $\sigma\sigma^x = \sigma\sigma^y$. Therefore, $\sigma^G \cap R = \{\sigma, \mu\}$ (with $\sigma \neq \mu$).

Without loss of generality we may, and shall, asume that $R \leqslant S$. If $R = S$, then $|\sigma^G \cap T| = 2$ and then, by [16, Theorem 3.3], $G \in \mathscr{L}$, a contradiction.

Thus $R \neq S$ and so, as $\sigma \in Z(R)$, $[S:R]=2$. Recalling that (see (5.9)) $R/C_R(P)$ is isomorphic to a subgroup of $GL(2,p)$, $|\mathscr{I}(C_R(P))|=1$ implies that $m(R) \leqslant 3$. Since $Z(S)$ is non-cyclic and $\sigma \notin Z(S)$, we see that $\Omega_1(Z(R)) = \Omega_1(R) = Z(S)\langle\sigma\rangle$ and $m(R)=3$. Thus $m(S) \leqslant 4$. If $m(S)=4$, then $S=AR$, where $A \in \mathfrak{A}_e(S)$, which yields $\sigma \in Z(S)$, a contradiction. Therefore, $m(S)=3$.

Let $\zeta \in \mathscr{C}(S)$ and set $C = C_G(\zeta)$ and $\bar{C} = C/O_{2'}(C)$. Also let $U \in \mathscr{X}^*(C)$ with $S \cap C \leqslant U$. Then $m(S)=3$ and Lemmas 3.1 and 4.3 imply that $E(\bar{C}) \cong A_7$ or $L_2(q)$ for some odd q and that $\Omega_1(C_{\bar{U}}(E(\bar{C}))) = \langle\zeta\rangle$. Employing (2.8) we see that $C_{\bar{U}}(E(\bar{C}))$ is isomorphic to either Q_8, $\mathbb{Z}_4$ or $\mathbb{Z}_2$.

By (5.13)(iii) and Glauberman's Z^*-theorem $H = C_G(\sigma)\, O_{2'}(H)$, and so $E(\bar{H}) \cong E(\overline{C_G(\sigma)})$, where $\bar{H} = H/O_{2'}(H)$. Suppose $\sigma \in \mathscr{C}(S)$ and set $E = E_{2'}(H)$ $(\neq 1)$. Since $\bar{E}$ is quasisimple, either $[E,\rho] \leqslant O_{2'}(E)$ or $[E,\rho]=E$. Suppose the latter possibility holds. Then, since ρ centralizes P and inverts $O_{p'}(F(H))$ we see that E centralizes $F(H) = F^*(H)$, which is impossible. Thus $[E,\rho] \leqslant O_{2'}(E)$ and hence $\langle\bar{\rho},\bar{\sigma}\rangle \leqslant C_{\bar{R}}(\bar{E})$, contradicting the fact that $m(S)=3$. So we conclude that $\sigma \in \mathscr{C}(S)$, and that H is 2-constrained. Also observe that since $\Omega_1(R) = \Omega_1(Z(R))$ is a strongly closed abelian 2-subgroup of H, $H = N_H(\Omega_1(R))\, O_{2'}(H)$ by (2.9).

Let $\zeta \in \mathscr{C}(S) \cap Z(R)$ and let $S \cap C \leqslant U \in \mathscr{X}^*(C)$, where $C = C_G(\zeta)$. Note that $R \leqslant U$. Since $\sigma \notin \mathscr{C}(S)$, $\bar{\sigma} \neq \bar{\zeta}$. Therefore, because $E(\bar{C}) \cong A_7$ or $L_2(q)$, q odd, $\bar{\sigma} \in E(\bar{C})\, C_{\bar{C}}(E(\bar{C}))$ would yield that $|\sigma^G \cap R| > 2$. Thus $E(\bar{C}) < E(\bar{C})\langle\bar{\sigma}\rangle \leqslant \operatorname{Aut} E(\bar{C})$, whence $E(\bar{C}) \cong L_2(q)$, $q \equiv 3, 5(8)$. (Note that the case $E(\bar{C})\langle\bar{\sigma}\rangle \cong S_7$ is excluded by $m(S)=3$.) Since $\langle\bar{\sigma}\rangle(\bar{U} \cap E(\bar{C})) \cong D_8$ and $m(S)=3$, we see (using (2.8)) that $U \cong V \times D_8$, where V is isomorphic to one of Q_8, $\mathbb{Z}_4$ and $\mathbb{Z}_2$.

First we consider the possibility that $\zeta \in Z(S)^G$. Then $S \cong V \times D_8$ where $V \cong Q_8$, $\mathbb{Z}_4$ or $\mathbb{Z}_2$. Whichever case holds we have that $\operatorname{Aut} S$ is a 2-group and so $S \in \operatorname{Syl}_2 G$ by (2.5). But then G cannot be a simple group, and so we conclude that $\zeta \notin Z(S)^G$. Therefore, $R = U$, which is contrary to $\Omega_1(R) \leqslant Z(R)$. Thus we have shown that $\mathscr{C}(S) \subseteq S \backslash R$.

Choose (as we may) $\zeta \in \mathscr{C}(S)$ such that $S_1 = C_S(\zeta) \in \mathscr{X}^*(C_G(\zeta))$. Set $C = C_G(\zeta)$ and $\bar{C} = C/O_{2'}(C)$. Then, since $\zeta \notin R = C_S(\sigma)$, $\Omega_1(Z(S_1)) = Z(S)\langle\zeta\rangle = \Omega_1(S_1)$. Hence $E(\bar{C}) \cong L_2(q)$, $q \equiv 3, 5(8)$. Since $\zeta \notin Z(S)^G$, Lemma 4.1(iii) gives $S_1 = \Omega_1(Z(S_1))$. We claim that $\overline{Z(S)} \leqslant E(\bar{C})$. Suppose this were false. Clearly $\overline{Z(S)} \cap E(\bar{C}) \neq 1$ and thus $\bar{S}_1 \backslash \langle\bar{\zeta}\rangle \subseteq \overline{Z(S)}^{\bar{C}}$ whence $S_1 \backslash \langle\zeta\rangle \subseteq Z(S)^G$. Thus, as $\zeta \notin Z(S)^G$, $\zeta \in Z^*(G)$ by the Glauberman Z^*-theorem, a contradiction. Therefore, $\overline{Z(S)} \leqslant E(\bar{C})$ as claimed, and so, in particular $Z(S)^{\#}$ fuses in G.

Recalling that $\mathscr{C}(S) \cap \Omega_1(Z(R)) = \varnothing$, we see that $O_{2'}$ is an $\Omega_1(Z(R))$-signalizer functor, and so $W = \langle O_{2'}(C_G(\tau)) \mid \tau \in \Omega_1(Z(R))^{\#}\rangle$ has odd order by [9]. By standard arguments we also have $(1 \neq)\, O_{2'}(H) \leqslant W$ and $N_G(R_1) \leqslant N_G(W) = N$ for all non-cyclic subgroups R_1 of R. Hence

$H=N_H(\Omega_1(Z(R)))\,O_{2'}(H)=N$. In particular $N_G(S)\leqslant N_G(Z(S))\leqslant H$. By (2.1)(v) $Z(S)^{\#}$ fuses in $N_G(S)$ and hence in H. However, $\rho\in Z(S)\cap C_R(P)$ and therefore $Z(S)\leqslant C_R(P)$, contradicting the previously obtained $|\mathscr{I}(C_R(P))|=1$. This contradiction arose from the assumption $E(H)=1$, and so we have proved that $E(H)\neq 1$.

(ii) Note that $O^p(F^*(H))\leqslant N_G(Y)\leqslant M$ and $[O_p(M), O^p(F^*(H))]=1$. Also, since $[P,\rho]=1$, $\rho\in M$. From (5.12)(v) we observe that ρ inverts $O_r(M)$ for all $r\in\pi(F(M))\backslash\{2,p\}$. Since $[E(H),\rho]=E(H)$ by (i), considering $\langle\rho\rangle E(H)\,O_r(M)$ we obtain $[E(H),O_r(M)]=1$. Thus $[E(H),O_{2'}(F(M))]=1$ and hence $[E(H),O_{2'}(M)]=1$, as required.

Combining (5.15)(iii) and Lemma 3.4(ii) gives $E(H)\leqslant E(M)$. Recall that (by Lemmas 5.4(a), (b) and (5.11)(ii)) $Z\in\mathfrak{U}_\sigma(H)$ for all $1\neq Z\trianglelefteq\trianglelefteq E(H)$. If $X=E(E(M)\cap H)\neq 1$, then $[X,F(H)]=1$ and so $1\neq[X,E(H)]$ is subnormal in both $E(M)$ and $E(H)$. Hence $F^*(M)\leqslant H$ and then $H=M$ by (5.9) and (2.14). So $E(E(M)\cap H)=1$. Let L be a component of $E(M)$. Suppose $L^\sigma\neq L$. Then $[O_p(H)\cap M,L]=1$ by (2.15)(ii). But $Z(O_p(H))\leqslant O_p(H)\cap M$ so contradicting $E(E(H)\cap M)=1$. Thus $L^\sigma=L$. Hence, using Lemma 3.4(ii) and $E(E(H)\cap M)=1$, we see that $E(H)$ and $E(M)$ have only one component and $E(H)<E(M)$. Since $[O_p(H)\cap M,E(M)]\neq 1$, $E(M)$ is isomorphic to either $L_2(q)$ $(q\not\equiv\pm 1(16)$, q odd, $q>3)$ or A_7 with $O_p(H)\cap M\leqslant E(M)\,C_M(E(M))$ by Lemma 3.4(iii). However Lemma 3.4(ii) demands that $E(H)$ is isomorphic to either $L_2(q)$ $(q=3,\ 5(8),\ q>3)$ or A_6. But then, since $[O_p(H)\cap M,E(H)]=1$, this gives $[O_p(H)\cap M,\ E(H))=1$. With this contradiction we have shown that either (i) or (ii) must hold.

If (i) does not hold, then by the above $R\cap O_2(H)\neq 1$, and so there exist $\mu\in\mathscr{I}(R\cap O_2(H))$ such that $[\mu,\sigma]=1$. Let $N\in\mathscr{M}(\mu)$. Because $\mu\in O_2(H)$, by (2.14), $[O_2(H),\ O_p(N)]=1$ and so $O_p(N)\leqslant H$. Since (5.9)–(5.12) is proved under the assumption that (i) does not hold and $\mu\in C_H(P)\cap C_G(\sigma)$, $H=N$ by (5.12)(iii). This completes the proof of Lemma 5.6.

LEMMA 5.7. *Either* (i) $Y\in\mathfrak{U}_\sigma(H)$ *for all* $1\neq Y=Y^\sigma\trianglelefteq\trianglelefteq F_0^*(H)$; *or*

(ii) $R\cap O_2(H)\neq 1$.

Proof. If $[E(H),\sigma]\neq 1$, then (i) holds by Lemmas 5.1 and 5.4(a), (b). So we may suppose $[E(H),\sigma]=1$. If $[O_{2'}(F(H)),\sigma]=1$, then Lemma 3.2 gives (ii). In the contrary case $[O_p(H),\sigma]\neq 1$ for some odd prime p and then the theorem follows from Lemma 5.6.

LEMMA 5.8. *Either* (i) $Y\in\mathfrak{U}_\sigma(H)$ *for all* $1\neq Y=Y^\sigma\trianglelefteq\trianglelefteq F_0^*(H)$; *or*

(ii) $m(O_{2'}(F(H)))\leqslant 2$.

Proof. Suppose the lemma is false. In view of Lemmas 5.6 and 5.7 we may suppose that $[\sigma,O_{2'}(F(H))]=1$ and $R\cap O_2(H)\neq 1$.

Let $\pi = \pi(F(H))$ and let W be a subgroup of $F^*(H)$ which is normalized by $\langle\sigma\rangle$. For M a subgroup of G containing $W\langle\sigma\rangle$ $\mathit{И}_M(W\langle\sigma\rangle)$ denotes the set of $W\langle\sigma\rangle$-invariant subgroups K of M such that $K = F^*(K) = O^{\pi}(K)$ and no component of K is contained in H.

Let V be an elementary abelian subgroup of $O_p(H)$, p odd, such that $m(V) \geqslant 3$ and $V \geqslant \Omega_1(Z(O_p(H)))$, and set $W = C_{F^*(H)}(V)$. Note that $\langle\sigma\rangle$ normalizes W. We now show that, under inclusion, $O^{\pi}(F(M))[E(M), \sigma]$ is the unique maximal element of $\mathit{И}_M(W\langle\sigma\rangle)$ where M is a proper subgroup of G containing $W\langle\sigma\rangle$. Let $K \in \mathit{И}_M(W\langle\sigma\rangle)$. Since $C_{F^*(H)}(W) \leqslant W \leqslant M$ and $\{2, p\} \subseteq \pi$, (2.14) implies that $[V, O_2(M)] = 1$ and $[O_2(H), O_q(M)] = 1$ for all $q \in \pi \backslash \{2\}$. Furthermore we also have that $[\sigma, O_q(M)] = O_q(M)$ for all $q \notin \pi$. From (2.13), (2.14) and the definition of $\mathit{И}_M(W\langle\sigma\rangle)$, $K = [K, \sigma] = [K, V] = [K, O_2(H)]$. Consequently $[K, O_q(M)] = 1$ for all $q \notin \pi$. Also we have $K = [K, V] \leqslant C_M(O_2(M))$ and $K = [K, O_2(H)] \leqslant C_M(O_q(M))$ for all $q \in \pi \backslash \{2\}$. So $[K, F(M)] = 1$. Since $\sigma \in M$, applying Lemma 3.4(iii) to M with $X = M \cap O_p(H)$ yields that V induces inner automorphisms on each component of M and therefore $V \leqslant E(M)\, C_M(E(M))$. Hence $K = [K, V] \leqslant E(M)\, C_M(E(M))$, which then yields $K \leqslant C_M(F^*(M))\, F^*(M)$. Therefore, $O^{\pi}(F(M))[E(M), \sigma]$ is the unique maximal element of $\mathit{И}_M(W\langle\sigma\rangle)$. Set $\mathit{И}(M) = O^{\pi}(F(M))[E(M), \sigma]$.

Suppose $1 \neq K \in \mathit{И}_G(W\langle\sigma\rangle)$. Using Lemma 3.4(iii) on $KW\langle\sigma\rangle$ gives that each component of K is normalized by V and centralized by some hyperplane of V. Thus there exists a hyperplane V_0 of V such that $\mathit{И}_0 = \mathit{И}(C_G(V_0)) \neq 1$. Let $v \in V^{\#}$ and set $\mathit{И} = \mathit{И}(C_G(v))$. Clearly $\mathit{И}_0 \leqslant \mathit{И}$. We will now show that $\mathit{И}_0 \trianglelefteq \mathit{И}$. Since σ inverts $F(\mathit{И})$ and $[\mathit{И}, \sigma] = \mathit{И}$, $F(\mathit{И}) \leqslant Z(\mathit{И})$ so it will suffice to prove that L normalizes $\mathit{И}_0$ where L is a component of $\mathit{И}$. Now $C_V(L)$ contains a hyperplane V_1 of V. If $V_1 = V_0$, then $L \leqslant \mathit{И}_0$. While $V_1 \neq V_0$ implies that $V = V_0 V_1$ and $\mathit{И}_0 = [\mathit{И}_0, V_1]$ whence, as $L \trianglelefteq \mathit{И}_0$, $[L, \mathit{И}_0] = 1$. Thus $\mathit{И}_0 \trianglelefteq \mathit{И}$.

Hence $\mathit{И}_1 = \mathit{И}_1(N_G(\mathit{И}_0)) \geqslant \langle \mathit{И}(C_G(v)) \mid v \in V_0^{\#} \rangle$. Since $m(V) \geqslant 3$, from Lemma 3.4(iii) it follows that $\mathit{И}_1$ is the unique maximal element of $\mathit{И}_G(W\langle\sigma\rangle)$. Set $W_1 = N_{F^*(H)}(W)$. Since $[\sigma, O_p(H)] = 1$, W_1 normalizes $W\langle\sigma\rangle$, hence W_1 normalizes $\mathit{И}_1$. So $\mathit{И}_1$ is the unique maximal element of $\mathit{И}_G(W_1\langle\sigma\rangle)$. Since $W \trianglelefteq \trianglelefteq F^*(H)$, we see that $\mathit{И}_1$ is the unique maximal element of $\mathit{И}_G(F^*(H)\langle\sigma\rangle)$. Therefore, as $C_G(\sigma)$ normalizes $F^*(H)\langle\sigma\rangle$, $C_G(\sigma) \leqslant N_G(\mathit{И}_1)$.

Let M be a maximal subgroup of G containing $N_G(\mathit{И}_1)$. Note that $F^*(H) \leqslant M$. Now $[\sigma, O_{2'}(F(H))] = 1$ and so, by the definition of $\mathscr{M}(\sigma)$, $C_{O_q(M)}(\sigma) \neq 1$ for all $q \in \pi(F(M)) \backslash \{2\}$. Hence $\pi(F(M)) \subseteq \pi(F(H))$ by (2.14). Then, since $|\pi(F^*(H)| > 1$, $F(M) \leqslant H$ and $[E(H), F(M)] = 1$ by (2.14). From Lemma 3.4(ii) and the definition of $\mathscr{M}(\sigma)$, we deduce that $E(H) = E(M) = 1$ and so $F^*(M) = F(M) \leqslant H$. Thus $H = M$ and so $\mathit{И}_1 = 1$, which contradicts $\mathit{И}_0 \neq 1$. Therefore we must have $\mathit{И}_G(W\langle\sigma\rangle) = \{1\}$.

Appealing to (3.14) gives that H is the unique maximal subgroup of G containing W. A final contradiction may now be obtained by arguing as in the last part of the proof of [10, (5.6)] and using Lemma 5.4(c).

Combining Lemmas 5.7 and 5.8 we obtain the main result of this section.

THEOREM 5.9. *Let $\sigma \in \mathscr{J}(S)$, $H \in \mathscr{M}(\sigma)$ and $R \in \mathscr{X}^*(H)$. Then either*

(i) *$Y \in \mathfrak{A}_\sigma(H)$ for all $1 \neq Y = Y^\sigma \trianglelefteq\trianglelefteq F_0^*(H)$; or*

(ii) *$R \cap O_2(H) \neq 1$ and $m(O_{2'}(F(H))) \leqslant 2$.*

6. THE p-CONSTRAINED CASE

As in the previous section we assume the following situation: $\sigma \in \mathscr{J}(S)$, $H \in \mathscr{M}(\sigma)$ and $R \in \mathscr{X}^*(H)$ such that $R \geqslant S \cap H$.

LEMMA 6.1. *H does not have any 2-components of type $SL(2, q)$, q odd $q > 3$.*

Proof. Suppose the lemma is false and argue for a contradiction. Let K be a 2-component of K such that $K/O_{2'}(K) \cong SL(2, q)$ where q is odd. Also J will be used to denote the product of all 2-components of H of type $SL(2, q)$. From Lemma 3.3(i), R normalizes each 2-component of J. Set $\bar{H} = H/O_{2'}(H)$. By (2.16), $\bar{R} \leqslant C_{\bar{H}}(\bar{K})\bar{K}$ and so, since, $Z(S)$ is non-cyclic and $Z(S) \leqslant R$, there exists $\tau \in Z(S)^\#$ such that $K = C_K(\tau)\, O_{2'}(K)$. Let $M \in \mathscr{M}(\tau)$. Set $\tilde{M} = M/O_{2'}(M)$ and $Z = E_{2'}(J \cap M)$. Note that $C_K(\tau)^\infty$ is a 2-component of Z and that $\sigma \in M$.

Since $O_{2'}(Z)$ is $C_M(\sigma)$-invariant, applying Lemma 3.4(i) in $\tilde{M}$ gives $[F(\widetilde{O_{2'}(Z)}), \tilde{\sigma}] = 1$, and so $[\widetilde{O_{2'}(Z)}, \tilde{\sigma}] = 1$. Thus $[\tilde{Z}, \tilde{\sigma}]$ is a product of components each isomorphic to $SL(2, q)$. Now $[\tilde{Z}, \tilde{\sigma}]$ is $C_{\tilde{M}}(\tilde{\sigma})$-invariant and so $[\tilde{Z}, \tilde{\sigma}] \trianglelefteq\trianglelefteq E(\tilde{M})$ by Lemma 3.4(i). If $\widetilde{C_K(\tau)^\infty} \leqslant [\tilde{Z}, \tilde{\sigma}]$, then $C_G(\tau)$ would have a 2-component isomorphic to $SL(2, q)$. On the other hand, $\widetilde{C_K(\tau)^\infty} \nleqslant [\tilde{Z}, \tilde{\sigma}]$ implies that $[C_K(\tau)^\infty, \sigma] \leqslant O_{2'}(C_K(\tau)^\infty)$ whence $[K, \sigma] \leqslant O_{2'}(K)$ and then $C_G(\sigma)$ has a 2-component isomorphic to $SL(2, q)$. Either possibility contradicts Lemma 4.3, and this completes the proof of the lemma.

LEMMA 6.2. *Let $\sigma \in \mathscr{J}(S)$ and $H \in \mathscr{M}(\sigma)$. Then neither of the following two situations can hold:*

(i) *For each elementary abelian subgroup A of S, there exists a hyperplane A_0 of A such that $\{H\} = \mathscr{M}(\sigma_0)$ for all $\sigma_0 \in A_0^\#$; and*

(ii) *suppose $m(S) \geqslant 5$ and for each elementary abelian subgroup A of*

S with $m(A) \geqslant 5$ there exists a hyperplane A_0 of A such that $\{H\} = \mathscr{M}(\sigma_0)$ for all $\sigma_0 \in A_0^{\#}$.

Proof. We suppose that either (i) or (ii) holds and show that a contradiction arises. First we establish that

(6.1) *$N_G(S_0) \leqslant H$ for all $S_0 \in \Sigma$.*

Let $S_0 \in \Sigma$, and suppose that (i) holds. If $m(S_0) \geqslant 3$, then we have $N_G(S_0) \leqslant H$. For let $A \in \mathfrak{A}_e(S_0)$ and let A_0 be the hyperplane of A as given in (i). Then for $n \in N_G(S_0)$, $A^n = B$ and $A_0^n \cap B_0 \neq 1$, where B_0 is the hyperplane of B as given in (i). Hence $H^n = H$, and so $N_G(S_0) \leqslant H$. While $m(S_0) \leqslant 2$ gives by the definition of Σ and Lemma 4.1(iii) that $\Omega_1(Z(S_0)) = Z(S)$, whence, by (2.3), $S \trianglelefteq N_G(S_0)$. Since $m(S) \geqslant 3$ by Lemma 4.1, we obtain $N_G(S_0) \leqslant N_G(S) \leqslant H$. Thus (6.1) holds in case (i).

Now suppose (ii) pertains. Let $S_0 \in \sum$ and set $N = N_G(S_0)$. As for (i) we see that $N_G(S_0) \leqslant H$ when $m(S_0) \geqslant 5$. So we must examine the situation $m(S_0) \leqslant 4$. Suppose $|S/S_0| = 2$. Then, since $m(S) \geqslant 5$ by hypothesis, $m(S_0) = 4$. Therefore, if $A \in \mathfrak{A}_e(S)$, then $A \cap S_0 \in \mathfrak{A}_e(S_0)$ and so $Z(S_0) \leqslant A \cap S_0 \leqslant Z(S)$. But then $S \trianglelefteq N$ by (2.3), and so $N \leqslant N_G(S) \leqslant H$. So we may assume $|S/S_0| > 2$. Thus, by (2.7)(ii) and [2], $\langle S^N \rangle / S_0$ is a non-soluble group with a strongly embedded subgroup. Since, by [20, Lemma 5.1(vii)] $\langle S^N \rangle / S_0$ must act faithfully upon $\Omega_1(Z(S_0))$, we see that $m(Z(S_0)) = 4$. Let $A \in \mathfrak{A}_e(S)$ and A_0 be as given in (ii). Then $A \cap S_0 = \Omega_1(Z(S_0))$ and so $m(A_0 \cap Z(S_0)) \geqslant 3$. Therefore, $N_G(S_0) \leqslant H$, and so we have (6.1).

From (6.1) and (2.7)

(6.2) *H controls fusion in S.*

Set $\bar{H} = H/O_{2'}(H)$.

(6.3) (i) $\bar{S} \leqslant \bar{E}(\bar{H}^*)$,

(ii) *$E(\bar{H}^*) = \bar{K}_1 \times \cdots \times \bar{K}_n$, where $\bar{K}_i \cong \bar{K}_j$ for all i, j and $\bar{K}_1 \in \mathscr{L}$.*

Let K denote the inverse image in H of $Z(\bar{S} \cap O_2(\bar{H}))$. Then, by (6.2), $R = S \cap K$ is a strongly closed abelian 2-subgroup of G and hence $R = 1$ by Lemma 4.1(i). Thus $\bar{S} \cap O_2(\bar{H}) = 1$ and so $E(\bar{H}) \neq 1$ by (2.3). From Lemma 3.1 we have $1 \neq \bar{S} \cap E(\bar{H}^*)$. Hence, by Lemma 4.1(i) and (6.2), $\bar{S} \leqslant E(\bar{H}^*)$ and $\bar{H}$ permutes the components of $E(\bar{H}^*)$ transitively. This proves (6.3).

Set $\bar{S}_i = \bar{S} \cap \bar{K}_i$, $i = 1, \ldots, n$, and let $A \in \mathfrak{A}_e(S)$. So $\bar{A} = \bar{A}_1 \times \cdots \times \bar{A}_n$ by (6.3), where $\bar{A}_i \in \mathfrak{A}_e(\bar{S}_i)$. Now A contains a hyperplane A_0 such that $\{H\} = \mathscr{M}(\sigma_0)$ for all $\sigma \in A_0^{\#}$. Suppose $\tau \in \mathscr{I}(Z(T) \cap S)$. So $\tau \in Z(S) \leqslant A$. Without loss of generality we may suppose $\bar{\tau} = \bar{\tau}_1 \cdots \bar{\tau}_m$, where each $\bar{\tau}_i \neq 1$

and $\bar{\tau}_i \in \bar{A}_i$, $i = 1,\ldots, m$. Since $[\bar{A}_i : \bar{A}_i \cap \bar{A}_0] \leqslant 2$, by (2.19) and (6.3)(ii), $\bar{\tau}_i$ is $\bar{K}_i$-conjugate to an element of $\bar{A}_i \cap \bar{A}_0$ whence $\bar{\tau}$ is $\bar{H}$-conjugate to an element of $\bar{A}_0$. Therefore τ is H-conjugate to an element of A_0, and so $C_G(\tau) \leqslant H$. Because of (6.2) we have the hypotheses of (2.20), which forces $G \in \mathscr{L}$. This is the desired contradiction, and so Lemma 6.2 holds.

LEMMA 6.3. *Let $\sigma \in \mathscr{I}(S)$ and $H \in \mathscr{M}(\sigma)$. Then $F^*(H)$ is not a p-group for any odd prime p.*

Proof. Supposing $F^*(H)$ is a p-group for some odd prime p we derive a contradiction.

(6.4) *Suppose $\tau \in \mathscr{I}(S)$, $K_\tau \in \mathscr{M}(\tau)$ and $F^*(K_\tau)$ is a p-group. If M is a maximal subgroup of G with $\tau \in M$ and $[F_0^*(K_\tau) \cap M, \tau] \neq 1$, then $F^*(M)$ is a p-group.*

Set $X = [F_0^*(K_\tau) \cap M, \tau]$. Since $F^*(K_\tau)$ is a p-group, $[F^*(K_\tau), \tau] \neq 1$ and so $X \in \mathfrak{A}(K_\tau)$ by Lemma 5.6. Hence, by (2.13)(v), $F^*(N_G(X))$ is a p-group. Using Lemma 3.4(i) gives that $X \leqslant O_p(M)$. Now $Z(O_p(M)) \leqslant N_{O_p(M)}(X) = Y$ and so $C_G(Y) \leqslant M$. Thus $O^p(F^*(M)) \leqslant F^*(C_G(Y))$. Because $F^*(N_G(X))$ is a p-group and $YC_G(Y) \leqslant N_G(X)$, $F^*(C_G(Y))$ is a p-group by (2.13)(v). Therefore, $O^p(F^*(M)) = 1$, and we have (6.4).

(6.5) *For $\tau \in \mathscr{I}(S)$ and $K_\tau \in \mathscr{M}(\tau)$, $F^*(K_\tau)$ is a p-group.*

Let $B = \langle \sigma \rangle Z(S)$. Then, as $F^*(H)$ is a p-group by hypothesis, $[F_0^*(H) \cap C_G(B_0), \sigma] \neq 1$ for some hyperplane B_0 of B. So $F^*(K_\rho)$ is a p-group for all $\rho \in B_0^\#$ and $K_\rho \in \mathscr{M}(\rho)$ by (6.4). In particular, $F^*(K_\mu)$ is a p-group for some $\mu \in Z(S)^\#$, $K_\mu \in \mathscr{M}(\mu)$ by Lemma 4.1(iii). Thus, if A is any maximal elementary abelian subgroup of S repeating the above argument using μ yields that A contains a hyperplane A_0 such that $F^*(K_\rho)$ is a p-group for all $\rho \in A_0^\#$ and $K_\rho \in \mathscr{M}(\rho)$.

By Glauberman's Z^*-theorem there exists $g \in G$ such that $\tau^g \in C_S(\tau) \backslash \{\tau\}$. Let $V = \langle \tau, \tau^g \rangle$ and let A be a maximal elementary abelian subgroup of S containing V, with A_0 as above. So $V \cap A_0 \neq 1$. If $\tau \in V \cap A_0$ or $\tau^g \in V \cap A_0$, then we are done. So we may suppose $V \cap A_0 = \langle \tau\tau^g \rangle$. Since $F^*(K)$ is a p-group where $K \in \mathscr{M}(\tau\tau^g)$ one of $[F_0^*(K) \cap C_G(\tau), \tau\tau^g] \neq 1$ and $[F_0^*(K) \cap C_G(\tau^g), \tau\tau^g] \neq 1$ must hold, and therefore (6.5) follows from (6.4).

(6.6) *Let $\mu \in Z(S)^\#$ and $N \in \mathscr{M}(\mu)$. If M is a maximal subgroup of G containing $SC_G(S)$ for which $[F_0^*(N) \cap M, \mu] \neq 1$, then $M = N$.*

Suppose (6.6) is false and choose $M \neq N$ such that $|M \cap N|$ is maximal. Set $J = M \cap N$ and $X = [F_0^*(N) \cap M, \mu]$. By (6.4) and (6.5), $F^*(N)$ and $F^*(M)$ are both p-groups. Since $\mu, X \leqslant J$, $1 \neq X \leqslant O_p(J)$ by Lemma 3.4(i).

By the maximal choice of $|M \cap N|$ either $N_M(O_p(J)) = J$ or $N_N(O_p(J)) = J$ and therefore $F^*(J)$ is a p-group by (2.13)(v).

We claim that M does not have any 2-components of type $SL(2, q)$, q a power of p, $q > 3$. Suppose this were false and let L be a 2-component of M such that $L/O_{2'}(L) \cong SL(2, q)$, q a power of p, $q > 3$. Since $SC_G(S) \leqslant M$, $L = \langle (S \cap L)^L \rangle$ by Lemma 3.1. As in Lemma 6.1, there exists $\rho \in Z(S)^{\#}$ such that $L = C_L(\rho)\, O_{2'}(L)$, and, by Lemma 3.3(i), $S \leqslant N_G(L)$. Let $D = C_L(\rho)^{\infty}$, $F \in \mathscr{M}(\rho)$ and $\bar{F} = F/O_{2'}(F)$. Note that $\bar{D}/O_{2'}(\bar{D}) \cong SL(2, q)$, that $\bar{D} \leqslant \langle \bar{S}^{\bar{F}} \rangle = \bar{F}^*$ and that $\bar{S}$ normalizes $\bar{D}$. So, by Lemma 3.7(ii), $[O_2(\bar{F}^*), \bar{D}] \leqslant O_2(\bar{F}^*) \cap \bar{D} \leqslant Z(\bar{D})$, and thus $[O_2(\bar{F}^*), \bar{D}] = 1$. Let $\bar{S}_0 = \bar{S} \cap E(\bar{F}^*)$. Then $[\bar{S}_0, \bar{D}] \leqslant E(\bar{F}^*) \cap \bar{D}$. Since $\bar{D} = \bar{D}^{\infty}$, using the 3-subgroups lemma, either $[\bar{S}_0, \bar{D}] = \bar{D}$ or $[\bar{S}_0, \bar{D}] \leqslant O_{2'}(\bar{D})$. Lemma 6.1 forbids $\bar{D} \leqslant E(\bar{F}^*)$, and so $\bar{D} = C_{\bar{D}}(\bar{S}_0)\, O_{2'}(\bar{D})$. Clearly $\bar{B} = C_{\bar{D}}(\bar{S}_0)$ normalizes each component $\bar{K}$ of $E(\bar{F}^*)$ and hence $\bar{B}^{\infty} \leqslant \bar{K} C_{\bar{F}^*}(\bar{K})$. Consequently $\bar{B}^{\infty}$ induces inner automorphisms upon $F^*(\bar{F}^*)$, which implies $\bar{B}^{\infty} \leqslant F^*(\bar{F}^*)$. Since $\bar{B}^{\infty}/O_{2'}(\bar{B}^{\infty}) \cong SL(2, q)$, this is contrary to Lemma 6.1, and so we have verified the above claim. Since $SC_G(S) \leqslant J$, an analogous argument shows that J too does not possess any 2-components of type $SL(2, q)$, q a power of p, $q > 3$.

Let $P \in И_N^*(S; p)$. Since $C_G(S) \leqslant N$, we have $P \in И_G(S; p)$. By Lemmas 3.9 and 6.1, $K_{\infty}(P) \trianglelefteq N$, which implies $P \in И_G^*(S; p)$. So $И_N^*(S; p) \subseteq И_G^*(S; p)$ and similarly, $И_M^*(S; p) \subseteq И_G^*(S; p)$.

Let $Q \in И_J^*(S; p)$. Again $Q \in И_G(S; p)$. From Lemma 3.9, $K_{\infty}(Q)) \trianglelefteq J$. Since, by the maximal choice of $|N \cap M|$, either $N_N(K_{\infty}(Q)) = J$ or $N_M(K_{\infty}(Q)) = J$ we see that $Q \in И_N^*(S; p) \cup И_M^*(S; p) \subseteq И_G^*(S; p)$. Employing Lemma 3.9 again forces $N = N_G(K_{\infty}(Q)) = M$. This completes the proof of (6.6).

(6.7) $\{K\} = \mathscr{M}(\mu)$ *for all* $\mu \in Z(S)^{\#}$.

By (6.5) and (6.6) we have that $|\mathscr{M}(\mu)| = 1$ for all $\mu \in Z(S)^{\#}$. Let $\eta, \tau \in Z(S)^{\#}$ and let $K_\tau \in \mathscr{M}(\tau)$, $K_\eta \in \mathscr{M}(\eta)$. To prove (6.7) we must show that $K_\tau = K_\eta$. Set $V = \langle \eta, \tau \rangle$. Since $F^*(K_\tau)$ is a p-group there exists $\rho \in V^{\#}$ such that $[F_0^*(K_\tau) \cap C_G(\rho), \tau] \neq 1$ and so $K_\tau \in \mathscr{M}(\rho)$ by (6.6). So either $K_\tau = K_\eta$ or $\rho = \eta\tau$. Similarly, arguing with η and τ reversed, we obtain either $K_\tau = K_\eta$ or $K_\eta \in \mathscr{M}(\eta\tau)$. Hence $K_\tau = K_\eta$, as required.

(6.8) *For any* $\tau \in \mathscr{I}(S)$ *define* $\theta_\tau = O_{2'}(C_G(\tau)) \cap O_{2'}(K)$. *Then for* $\tau, \mu \in \mathscr{I}(S)$ *with* $[\tau, \mu] = 1$, $\theta_\tau \cap C_G(\mu) \leqslant \theta_\mu$.

Set $C = C_G(\mu)$, $\bar{C} = C/O_{2'}(C)$ and $X = \theta_\tau \cap C_G(\mu)$. Suppose $X \not\leqslant \theta_\mu$. Then $X \not\leqslant O_{2'}(C)$. So by (2.13)(vi) there exists a $\langle \bar{\tau}, \bar{X} \rangle$-invariant component of $E(\bar{C})$ which is not centralized by either $\bar{\tau}$ of $\bar{X}$. Let $L \leqslant C$ be such that $\bar{L}$ is such a component. If $L \leqslant K$, then $[\bar{X}, \bar{L}] = 1$. So $L \not\leqslant K$. By Lemma 4.1(iii) and (6.7), $O_{2'}(C) \leqslant K$. Thus $\bar{L} \nleqslant \langle C_{\bar{C}}(\bar{\zeta}) \mid \bar{\zeta} \in \overline{Z(S)}^{\#}\rangle$ and so $\overline{Z(S)} \leqslant N_{\bar{C}}(\bar{L})$ by (2.15)(iv). Moreover, from (2.17)(v), $\bar{L}$ can only be of type $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$, $L_3(2^n)$ or $PSp_4(2^n)$, and then, by (2.17)(iv), the only possibility is $\bar{L} \cong L_2(4)$ with $\bar{L}\langle\bar{\tau}\rangle \cong S_5$. Consider $\bar{J} = \bar{L}\langle\bar{\tau}\rangle\overline{Z(S)}$. Then $\bar{J}/C_{\bar{J}}(\bar{L}) \cong S_5$. Since $L \nleqslant K$, $\overline{Z(S)} \cap C_{\bar{J}}(\bar{L}) = 1$. Because $Z(S)$ is non-cyclic and $\bar{J}/C_{\bar{J}}(L)$ has dihedral Sylow 2-subgroups, there exists $\bar{\mu}_1, \bar{\mu}_2 \in \overline{Z(S)}^{\#}$ such that $\bar{\mu}_1 = \bar{\tau}(\text{mod } C_{\bar{J}}(\bar{L}))$ and $\bar{\mu}_2 \in \bar{L}(\text{mod } C_{\bar{J}}(\bar{L}))$. Hence $\bar{L} = \langle C_{\bar{L}}(\bar{\zeta}) \mid \bar{\zeta} \in \overline{Z(S)}^{\#}\rangle$, which has been ruled out. Therefore, we conclude that $X \leqslant \theta_\mu$, as required.

Observe that, because of Lemma 4.1(iii) and (6.7), $\langle \theta_\tau \mid \tau \in \mathscr{I}(S)\rangle = O_{2'}(K)$. Since $\mathrm{SCN}_3(S) \neq \varnothing$, (6.8) and a signalizer functor type argument gives $N_G(S_0) \leqslant N_G(O_{2'}(K)) = K$ for all elementary abelian 2-subgroups S_0 with $m(S_0) \geqslant 2$. Thus, by (2.7), K controls fusion in S and then, combining (2.20) and (6.7), we obtain $G \in \mathscr{L}$, a contradiction. This proves Lemma 6.3.

7. Some Reductions

In this section, by combining the results of the two preceding sections, we obtain some important restrictions upon G.

Lemma 7.1. *Let $\sigma \in \mathscr{I}(S)$ and $H \in \mathscr{M}(\sigma)$. Suppose p is an odd prime such that $O_p(H)$ is not cyclic. Let $\sigma_1 \in \mathscr{I}(S)$ be such that $[\sigma, \sigma_1] = 1$, let $H_1 \in \mathscr{M}(\sigma_1)\backslash\{H\}$ and assume that the following hold:*

(a) *either $[O_p(H), \sigma] \neq 1$ or $m(O_p(H)) \geqslant 3$; and*

(b) $Q = O_p(F_0^*(H)) \cap H_1 \neq 1$.

Then

(i) $Q \cap O_p(H_1) = 1 = [Q, \sigma]$; *and*

(ii) *Q is cyclic.*

Proof. Set $P = O_p(H)$ and $P_1 = O_p(H_1)$. Supposing that neither (i) nor (ii) holds we obtain a contradiction, as in [10, (6.3)], with the aid of Lemmas 3.4 and 3.5 and Theorems 5.9 and 6.3. Because Lemma 3.5 is not the exact analogue of [10, (3.12)] we must also eliminate the following situation: $m(P_1) \leqslant 2$, $[Q, O^p(F^*(H_1))] = 1$ and $Q \nleqslant P_1$. Set $K = H_1^* O_{2'}(H_1)$ and $\tilde{K} = K/C_K(P_1)$. From Lemma 3.5 we have that $(\tilde{K})^\infty \cong SL(2, p)$, $p > 3$ and that $\langle Q^K\rangle C_K(P_1) \geqslant (\tilde{K})^\infty$. Note that $\langle Q^K\rangle \leqslant C_K(O^p(F^*(H_1)))$. Let J be a minimal normal subgroup of K contained in $C_K(O^p(F^*(H_1)))$ minimal subject to it covering $(\tilde{K})^\infty$. Since $C_J(P_1) \leqslant Z(F(H_1))$ by (2.13), $J^\infty = J$ and the Schur multiplier of $SL(2, p)$ is trivial, we see that $C_J(P_1) \leqslant Z(O_p(H_1))$. Thus $J/O_p(J) \cong SL(2, p)$ and so J is a 2-component of H_1 of type $SL(2, p)$, $p > 3$, which is impossible by Lemma 6.1.

Therefore, Lemma 7.1 holds.

LEMMA 7.2. *Let $\sigma \in \mathscr{I}(S)$, $H \in \mathscr{M}(\sigma)$ and $p \in \pi(F(H))\backslash\{2\}$. Then either*

(i) *$O_p(H)$ is cyclic; or*

(ii) *there exists $\sigma_0 \in \mathscr{I}(S)$ such that $H \in \mathscr{M}(\sigma_0)$ and $[O_p(H), \sigma_0] = 1$.*

Further, if $\sigma \neq \sigma_0$, then $\sigma_0 \in Z(S)$ and $\{H\} = \mathscr{M}(\sigma_0)$.

Proof. Suppose $O_p(H)$ is non-cyclic and $[O_p(H), \sigma] \neq 1$, and let A be a maximal elementary abelian subgroup of S containing σ. Then $[O_p(F_0^*(H)), \sigma] \neq 1$ and so there exists a hyperplane A_0 of A such that $[O_p^*(F_0(H)) \cap C_G(A_0), \sigma] \neq 1$. Hence $\{H\} = \mathscr{M}(\sigma_0)$ for all $\sigma_0 \in A_0^{\#}$ by Lemma 7.1. Since $Z(S) \leqslant A$, $Z(S) \cap A_0 \neq 1$ by Lemma 4.1(iii). If $[O_p(H), \sigma_1] \neq 1$, where $\sigma_1 \in Z(S) \cap A_0$, then repeating the above argument yields that each maximal elementary abelian subgroup B of S contains a hyperplane B_0 such that $\{H\} = \mathscr{M}(\rho)$ for all $\rho \in B_0^{\#}$. Lemma 6.2(i) forbids such a situation, and so we have proved the lemma.

LEMMA 7.3. (i) *Let A be an elementary abelian subgroup of S with $m(A) \geqslant 5$ and let $\sigma \in A^{\#}$. Then $m(O_{2'}(F(H))) \leqslant 2$ for each $H \in \mathscr{M}(\sigma)$.*

(ii) *Either $m(O_{2'}(F(H)) \leqslant 2$ for all $H \in \mathscr{M}(\sigma)$, $\sigma \in A^{\#}$ and $A \in \mathfrak{A}_e(S)$ or $m(S) = 3$.*

Proof. Suppose A is a non-cyclic elementary abelian subgroup of S and $\sigma \in A^{\#}$. Let $H \in \mathscr{M}(\sigma)$ and p be an odd prime such that $m(P) \geqslant 3$, where $P = O_p(H)$. Then $m(O_p(F_0^*(H))) \geqslant 2$. Let P_0 be a minimal A-invariant non-cyclic subgroup of $O_p(F_0^*(H))$. Since A is non-cyclic $m(P_0/\phi(P_0)) = 2$, and so $m(A/A_0) \leqslant 2$, where $A_0 = C_A(P_0)$. By Lemma 7.1 we have

(7.1) $\{H\} = \mathscr{M}(\sigma_0)$ *for all* $\sigma_0 \in A_0^{\#}$.

We now prove part (i). So A is an elementary abelian subgroup of S with $m(A) \geqslant 5$ and we may assume that $Z(S) \leqslant A$. Suppose (i) does not hold. Thus the above holds and $m(A_0) \geqslant 3$. Clearly there exists $A_1 < A$ such that $[A : A_1] = 2$, $A_0 \leqslant A_1$ and $Q = C_{P_0}(A_1) \neq 1$. Let $\sigma_1 \in A_1^{\#}$ and $H_1 \in \mathscr{M}(\sigma_1)$. From (7.1), $P_0 \leqslant \bigcap_{\sigma_0 \in A_0^{\#}} O_p(C_G(\sigma_0))$ and then Lemma 3.6 yields that $1 \neq Q \leqslant O_p(H_1) \cap P_0$, which in turn gives $H_1 = H$ by Lemma 7.1. So $\{H\} = \mathscr{M}(\sigma_1)$ for all $\sigma_1 \in A_1^{\#}$. Since $1 \neq \sigma^* \in Z(S) \cap A_1$, we may repeat the above arguments with σ^* in place of σ to obtain the situation depicted in Lemma 6.2(ii), and this gives the desired contradiction. Thus (i) holds.

If $m(S) \geqslant 5$, then (ii) follows from part (i). So in proving (ii) we may suppose $m(S) \leqslant 4$. Let $A \in \mathfrak{A}_e(S)$, $\sigma \in A^{\#}$ and $H \in \mathscr{M}(\sigma)$ be such that $m(O_p(H)) \geqslant 3$ for some odd prime p. Supposing $m(S) = 4$ we seek a contradiction. Thus A_0 is non-cyclic

(7.2) *If* $\sigma_0 \in A_0^{\#}$ *and* $\sigma_0^g \in A$, $g \in G$, *then* $g \in H$.

By (7.1), $\{H\} = \mathscr{M}(\sigma)$. So $H^g \in \mathscr{M}(\sigma_0^g)$ and $m(O_p(H^g)) \geqslant 3$. Since $O_p(H^g) \leqslant \langle C_G(\sigma^*) \mid \sigma^* \in A_0^{\#} \rangle \leqslant H$, we see that $H^g = H$ by Lemma 7.1, and so $g \in H$.

Therefore, by (7.2), $S \leqslant N_G(A) \leqslant H$. Since P is non-cyclic, Lemma 7.2 implies $C_S(P) \neq 1$ and so, in particular, $1 \neq \sigma_1 \in Z(S) \cap A_0$. Consequently $N_G(Z(S)) \leqslant H$ by (7.2). Further, for $B \in \mathfrak{U}_e(S)$, we have $\sigma_1 \in B$ and $\{H\} = \mathscr{M}(\sigma_1)$, and so (7.2) holds for B. Hence $N_G(B) \leqslant H$ for all $B \in \mathfrak{U}_e(S)$ and therefore $N_G(S_0) \leqslant H$ for all $S_0 \in \Sigma$ (see Lemma 6.2(ii)).

So H control fusion in S by (2.7) and hence $\bar{S} \leqslant E(\bar{H})$ ($\bar{H} = H/O_{2'}(H)$). Moreover, because of Lemma 6.1, either $E(\bar{H}) = \bar{K}_1$ or $E(\bar{H}) = \bar{K}_1 \times \bar{K}_2$ where $\bar{K}_1$ is simple and, in the latter case $\bar{K}_1 \cong \bar{K}_2$. Since S is not abelian, $S \in \mathrm{Syl}_2\, E(\bar{H})$, and so we see that $\bar{K}_1$ does not possess a non-trivial strongly closed abelian 2-subgroup. Thus, if $E(\bar{H}) = \bar{K}_1 \times \bar{K}_2$, then $S \cong D_8 \times D_8$, which implies that $N_G(S)/C_G(S)$ is a 2-group. So $S \in \mathrm{Syl}_2\, G$ by (2.5), contrary to the simplicity of G. Therefore, we must have $E(\bar{H})$ simple. Set $E = E_{2'}(H)$, and let q be an odd prime. If $O_q(H)$ were cyclic; then $[E, O_q(H)] = 1$. While, if $O_q(H)$ is non-cyclic, then, by Lemma 7.2, there exists $\sigma^* \in \mathscr{I}(S)$ such that $[\sigma^*, O_q(H)] = 1$. Since $\bar{S} \leqslant E(\bar{H})$ and $E(\bar{H})$ is simple, $[E(\bar{H}), \bar{\sigma}^*] = E(\bar{H})$ whence $[E, \sigma^*] = E$. Thus $[E, O_q(H)] = 1$ also. Hence $[E, O_{2'}(F(H))] = 1$ and so $E = E(H)$. In particular $[S, P] = 1$, which by Lemma 7.1 gives, since $H \in \mathscr{M}(\sigma_1)$ for some $\sigma_1 \in Z(S)^{\#}$, $\langle C_G(\sigma^*) \mid \sigma^* \in \mathscr{I}(S) \rangle \leqslant H$. This contradicts Lemma 4.1(iv) and completes the proof of part (ii).

Acknowledgments

I would like to thank Professor George Glauberman for his interest in this work and Professor Helmut Bender for making a crucial observation, which is used in Lemma 5.8.

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